Statistica Applicata - Italian Journal of Applied Statistics Vol. 34 (3)
doi.org/10.26398/IJAS.0034-015

CONNECTED ASPECTS OF RISK MEASURE AND INEQUALITY INDICES WITH RELIABIITY CONCEPTS

Zahra Behdani[1]
Department of Statistics Behbahan Khatam Alanbia University of Technology, Behbahan, Iran.

Gholam Reza Mohtashami Borzadaran
Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Iran

Abstract *One of the commonalities of the theory of reliability, economics and insurance is the study of non-negative data, generally skewed (longevity, profit and income), fitting the appropriate model to this data and finding their characteristics. Risk measures play a serious role in the economics of insurance and in the most financial institution. The concepts of inequality measures are an important effect in the evaluation of the inequality of income distributions and wealth in economic, social sciences, and other areas.*
This paper states some properties of inequality indices and risk measures and applications also, compare them with each other. The main aim of this paper is to investigate the relationship between some inequality measures and risk measures with reliability concepts. Specially, we are interested in finding the connection between Lorenz ordering and risk measures. Examining the relationship between inequality indicators, risk and reliability measures allows the researcher to use the criteria of each of these concepts to examine the other concept.

Keywords: *Risk measures, Inequality indices, Order statistics, Lorenz curve, Value-at-Risk, Conditional Value-at-Risk.*

1. INTRODUCTION

Usually, when it comes to types of peril, the word risk is highlighted; but we need to know that risk is not just a hazard and a negative concept. Risk in the general sense indicates uncertainty about the future. We may put ourselves at risk and face positive opportunities. These positive opportunities can be explained in financial terms. Therefore, risk has two positive and negative dimensions. The concept of risk in finance and investment is generally considered. Also, risk can be defined as an act or event whose outcome is not known. In the definition, as it is clear, it is not necessarily a bad situation for human beings, but it can also be a messenger

[1]Zahra Behdani, behdani@bkatu.ac.ir

of profit and benefit; so risk has two dimensions which are called bad risk or good risk and chance. Here, the state of non-change in circumstances must also be considered. To make the risk clearer, we can consider the game stock market, which enters the stock market to make a profit and invests in certain stocks. The stock market is facing a state of uncertainty here; Uncertainty about rising stock prices and gains or declining stock values and losses, or an unchanged vision, so it should be added that the result of open stock risk is either a profit or a loss or a state of change.

The increasing complexity and uncertainty of the current economic system suggests that many of the problems involve decisions under uncertainty and hence with unknown consequences. Although risk control has been a central issue in financial optimization since the original work at Markowitz in 1952, it is only in recent decades that decision makers working in other areas of the program become aware of the importance of risk analysis and control. In financial programs, risk-taking or scarcity measures are increasingly being considered. Some of the most important of these measures are Value-at-Risk and Value at Risk Condition (CVaR).

Inequality refers to the measurement of imbalance or unequal distribution in a system, which may be social, economic, political, diversity, etc. In economics, it refers to how economic metrics are distributed among individuals in a group, among groups in a set of population, or among countries. Economists generally reckon about three broad areas of economic disparity. They are with respect to wealth, known as wealth inequality, income or income inequality and consumption or consumption inequality. Inequality of outcome from economic transactions occurs when some individuals gain much more than others from an economic transaction. Inequality of opportunity occurs when individuals are denied access to institutions or employment, which limits their ability to benefit from living in a market economy.

Inequality index of income distribution provides a useful tool for analysis and based. It is possible to evaluate fiscal policies and compare the distribution of income between different societies and times. Fairly recently, several inequality curves have been made or investigated as the descriptors of income inequality. The Lorenz curve was first introduced in 1905. The Bonferroni curve and the Zenga-2007 curve are main the functions of the Lorenz curve. Arnold (Arnold, 2015) proved, they each specify the parent distribution up to scale factor, and they each, give an inequality partial order that is corresponding to the Lorenz order. The Lorenz curve is inspected as a very advantageous tool of economic suitable to its important role in the evaluation of the inequality of income distributions and wealth.

Reliability is a broad concept. It is applied whenever we expect something to behave in a certain way. It is one of the metrics that are used to measure quality. The notion of reliability, in the statistical sense, is the probability that an equipment or unit will perform the required function under the conditions specified for its operations for a given period of time. The primary concern in reliability theory is to understand the patterns in which failures occur, for different mechanisms and under varying operating environments, as a function of its age. This is accomplished by identifying the probability distribution of the lifetime represented by a non-negative random variable. Accordingly, several concepts have been developed that help in evaluating the effect of age, based on the distribution function of the lifetime random variable and the residual life X. Concepts of aging describe how a component or a system improves or deteriorates with age; and they are very serious in the reliability analysis. In reliability, several aging classes of life distributions have been presented to explain the various forms of aging. Different order relations have been developed using measures in connection with many fields such as reliability, economics, queuing theory, survival analysis, insurance, operations research, etc. distribution function frame work.

The relationship between inequality and reliability indicators in different ways has already been investigated in several articles. For example, Zenga curve shall be interpreted as the difference in average age of components which has survived beyond age X from those which has failed before attaining age X, expressed in terms of average age of components exceeding age X.

Several authors have discussed associations between income and wealth inequality measures and some main concepts applied in survival analysis and reliability theory. Chandra and Singpurwalla (Chandra and Singpurwalla, 1981) explained the relationship between Lorenz curve and Gini index in economics and total time on test and mean residual life in reliability. Klefsjo (Klefsjö, 1984) produced more effects along the exact same lines and shown reliability understandings of more methods from economics. Giorgi and Crescenzi (Giorgi and Crescienzi, 2001) demonstrated some associations between the total time on test and Bonferroni curve and showed how the Bonferroni curve might be used in reliability theory. Singpurwalla (Singpurwalla, 2007) observed the connection involving the survival function in reliability and the advantage pricing method of fixed income tool like a risk-free zero discount bond. (Behdani et al., 2018, 2019) considered the relationships between the weighted distributions, generalized failure rate and generalized reversed failure rate with some inequality measures. They also studied some properties of double truncated distributions and in view of income inequality (Behdani et al., 2020).

Today, one of the important goals of researchers is to establish a connection between different sciences or various branches of each science. Some indicators of inequality can be generalized to non-income distribution phenomena. In the present work, we have examined the connection between the existing inequalities measures risk measures and reliability concepts, the relationship of the concepts of inequality indices with certain reliability concepts are exploited to obtain characterization results for probability distributions. Further some results on a stochastic order using inequality curves are also established. There are many reasons to study the relationship between the risk measures, the reliability concepts and the inequality measures. For example, the study of the relationship between inequality indices, risk measures and measurement standards of reliability makes it possible to use each of these three concepts for studying the other one. We are able to set some other criteria for exponentiation based on the Lorenz curve and Gini index to use insurance and reliability. Also, we can obtain some new properties for the Lorenz curve, the Gini index, risk measures and the reliability indices. Moreover, we are able to indicate that the Lorenz curve can be expended to create a variant explanation of lifetime and insurance data and vice versa and, as well, determine the bound of the class of lifetime distributions in terms of its Lorenz curve, risk measures and the other index inequalities or risk measure. The purpose of this article is to find the relationship between indicators of economic inequality and some concepts and indicators of reliability and insurance. In fact, in this text, we seek to find some connection between insurance economics and reliability, so that by having the indicators of each, we can comment on other features and characteristics. Finding a connection between the concepts of reliability and indicators of economic inequality can be very useful. Because finding the connection between these concepts enables researchers to use the results obtained for each of these concepts and indicators separately. For example, the Gini coefficient can be used for exponential tests, or a new indicator of inequality can be defined using the concepts of reliability, or interpretations of economic inequality can be obtained for lifetime data. For example, the Zenga curve can be used as the difference between the average lifespan of components that were destroyed before reaching the desired age and the average age of those who have reached the desired age compared to those who have reached the desired age. Interpreted. In addition, categories of lifetime distributions can be obtained using inequality indices and Insurance. (Righi, 2019) presented a composition of risk and deviation measures, which contemplate these two concepts. Based on the proposed Limitedness axiom, they proved that this resulting composition, based on properties of the two components, is a coherent risk measure. Similar results for the cases of convex and co-monotone risk measures are exposed. Using the Gini coefficient,(Eugene et al., 2021) introduced and explored Gini Shortfall (GS), a more comprehensive risk measure than VaR and ES. GS provides information on the variability of data in distribution tails measured with Tail Gini functional, a tail variability measure based on the variability measure Gini Mean Difference or Gini functional. Using an improper probability measure can affect risk forecasting and lead to wrong financial decisions. (Berkhouch et al., 2022) proposed a Deviation-based approach for quantifying model risk associated with choosing an in appropriate probability measure for risk forecasting. This measuring approach provides us with information about how far our risk measurement process could be affected by model risk. They introduced the Gini coefficient as one of the criteria for measuring deviation. (Berkhouch et al., 2018) introduced a risk measure that extends the Gini-type measures of risk and variability, the Extended Gini Shortfall, by taking risk aversion into consideration. (Furman et al., 2017) introduced and explored Gini-type measures of risk and variability, and developed the corresponding economic capital allocation rules. The content of this paper are as follows. The Lorenz curve and several inequality measures are studied in Sections 2. Risk measures and theire properties are discussed in Section 3. In Section 4, we give a brief review of the reliability consepts and the stochastic order. Finally, in Section 5 we study relations between the inequality measures and the risk measures.

2. INEQUALITY INDICES

Economists are interested in measuring how incomes and wealth are distributed in societies. This interest is due to the effect of how income is distributed on different economic categories. Researchers in the field of income distribution agree that higher average incomes increase social welfare, while higher inequality reduces social welfare. The question is, how do we measure inequality ?

In one of the oldest topics, political economy is of fundamental importance. It means how income from production is divided among the factors of production or how much each economic sector contributes. Unequal distribution of factors of production will naturally lead to an inadequate distribution of income. Therefore, in order to justify the distribution of income, the factors of production must be distributed fairly and equally among different individuals and enterprises before production. Many indices have been introduced to measure income inequality. Among them, the Lorenz curve, Gini coefficient, Bonferroni curve and Zenga curve can be named.

The Lorenz curve was first defined by Lorenz (1905) ((Lorenz, 1905)) and presents a graphical tool to investigate income inequality for about one hundred years since they were designed. [2]

Definition 2.1. The Lorenz curve of X, a non-negative random variable with positive and finite mean, is given by

$$L(p) = \frac{1}{E(X)} \int_0^{F^{-1}(p)} u f(u) du, \quad 0 \leq p \leq 1. \tag{2.1}$$

It is worthwhile to be mentioned that L_X has the following properties:

- Lorenz curve is a distribution function, twice differentiable, convex, increasing.

- $L(0) = 0$ and $L(1) = 1$ on $[0;1]$.

- $lim_{p \to 1} L'(p)(1-p) = 0$, $L_X(p) \leq p$.

Let the new random variable $X_{a,b}$ defined as the distribution of X truncated to the closed interval $[a,b]$. Behdani et al. (Behdani et al., 2020) introduced the Lorenz curve for a double truncated random variable ($X_{a,b}$) as follows:

$$L_{X_{a,b}}(p) = \frac{L_X(p[F(b) - F(a)] + F(a)) - L_X(F(a))}{L_X(F(b)) - L_X(F(a))} \tag{2.2}$$

Left and right truncated are a special case of doubly truncated when $b \to \infty$ and $a \to 0$. Suppose $p = F(x)$ is the ratio of people whose income is less or equal x.

[2] Let X be a non-negative continuous random variable with positive and finite mean. We propound $F_X(u) = P\{X \leq u\}$ for the distribution function and apply the symbolisms f to score respective probability (density) distributions. When no confusion may occur, we write simply F instead of F_X. Let $F_X^{-1}(u)$ be its right continuous inverse, i.e. $F_X^{-1}(u) = \inf\{v : F_X(v) \geq u, u \in (0,1)\}$, where the lower and upper bounds of the support of F_X (S_F) are $F_X^{-1}(0)$ and $F_X^{-1}(1)$ respectively. As F_X^{-1} is non-decreasing, it is continuous everywhere, except on an at most countable set of points.

In this case, the mathematical expectation (average) income of these people can be shown as:

$$E[X|X \leq x] = \frac{\mu L(p)}{p} \tag{2.3}$$

And therefore

$$E[X|X > x] = \frac{\mu(1 - L(p))}{1 - p} \tag{2.4}$$

The Gini coefficient has been found helpful to analysis the inequality of incomes. The value of the Gini coefficient shows the extent of income inequality. The Gini coefficient is the most famous criterion for income inequality. This is corresponding to twice the region between the equality line and the Lorenz curve, which is exactly a relative measure of income inequality:

$$G = 2\int_0^1 (y - L(y))dy = 1 - 2\int_0^1 L(y)dy. \tag{2.5}$$

A relatively minor adjustment of the Lorenz curve is the Bonferroni curve $B_X(p)$. It was written as:

$$B_X(p) = \frac{E(X|X \leq F^{-1}(p))}{\mu} = \frac{L(p)}{p}, 0 < p \leq 1. \tag{2.6}$$

Consequently, we cannot say in general that Bonferroni curve starts from the origin of the orthogonal plane, as it depends on the definition of X ((Giorgi and Crescienzi, 2001)). The Bonferroni curve is could be concave in some parts and convex in the others and strictly increasing. The Bonferroni curve has uses in fields such as medicine, insurance, demography, and reliability. We can rewrite $B_X(p)$ as $1 - \dfrac{E\{X|X \leq F^{-1}(p)\}}{\mu}$.

The Zenga index is the another measure of income inequality. The ratio of the mean income of the poorest $100p$ in the distribution to that of the rest of the distribution, namely the $100(1-p)$ richest is the Zenga curve $Z(p)$. The Zenga inequality measure is defined as:

$$Z(p) = 1 - \frac{E(X|X \leq F^{-1}(p))}{E(X|X > F^{-1}(p))} \tag{2.7}$$

The Zenga curve can be written as:

$$Z_X(p) = 1 - \frac{L(p)}{p} \cdot \frac{1 - p}{1 - L(p)} \quad p \in (0, 1). \tag{2.8}$$

X is smaller than Y in the Lorenz order ($X \leq_L Y$), Bonferroni order ($X \leq_B Y$) or Zenga order ($X \leq_Z Y$) iff $L_Y(p) \leq L_X(p)$, $B_Y(p) \leq B_X(p)$ or $Z_X(p) \leq Z_Y(p)$ respectively. These orders are invariant with respect to scale transformation. The Lorenz order is the natural mathematical abstraction of Lorenz's (1905) comparison of income distributions via nested Lorenz curves ((Arnold, 2012)). From definitions of the Zenga and Bonferroni curves and definitions of the Zenga, Bonferroni and Lorenz orders immediately conclude that

$$X \leq_L Y \Longleftrightarrow X \leq_B Y \Longleftrightarrow X \leq_Z Y. \tag{2.9}$$

The excess wealth measures EW_X is closely related to the Lorenz curve and it defined as

$$\begin{aligned} EW_X(p) &= \int_{F^{-1}(p)}^{\infty} \overline{F}(x)dx, \\ &= \int_p^1 (F^{-1}(q) - F^{-1}(p))dq, \\ &= \mu(1 - L(p)) - F^{-1}(p)(1-p) \end{aligned} \tag{2.10}$$

The excess wealth function is equal to the stop-loss function evaluated at $F^{-1}(p)$ and, therefore it is the net premium for a stop-loss contract with fixed retention $x = F^{-1}(p)$ ((Belzunce et al., 2015)).

3. RISK MEASURES

The probability that the actual return on investment will deviate from its projected return is called risk. Risk also includes the possibility of losing all or part of the investment principal. There are different criteria for measuring investment risk. Risk measurement criteria were first determined by studying statistical dispersion indices and then newer measures were developed. Risk is one of the first concerns of investors and as an important criterion in decision making It is considered an investment, and to accept an investment without considering this criterion is to be in a harmful situation. In fact, successful investors are those who accept an acceptable level of risk, because uncertainty does not always mean a detrimental future. If the conditions of uncertainty are such that they shape a positive future, then it would be reasonable to take risks. In the stock market, for example, uncertainty does not always mean lower stock prices.

3.1. DIFFERENCE BETWEEN RISK AND UNCERTAINTY

Risk and uncertainty are two important terms in the world of finance and business. Although some tend to use these two terms interchangeably, there is a distinct difference between risk and uncertainty. Risk is the chance that an investments actual outcome will differ from the expected outcome, while uncertainty is the lack of certainty about an event.The main difference between risk and uncertainty is that risk is measurable while uncertainty is not measurable or predictable. Other differences between the two include the following:

- Risk is the chance that an investments actual outcome will differ from the expected outcome, while uncertainty is the lack of certainty about an event.
- In risk, potential outcomes are known, but in uncertainty, potential outcomes are unknown.
- Risks can be measured and quantified using theoretical models, but uncertainty cannot be measured.
- Moreover, risks can be controlled if proper measures are taken at the right time; however, uncertainty is beyond control.

Therefore, the concept of risk plays a key role in financial markets and therefore the identification of risk types, measurement and management is of great importance. To read more about the difference between risk and uncertainty, see (Gifford, 2003). Denuit ((Denuit et al., 2006)) introduces the risk in following definition.

Definition 3.1. A risk R is a non-negative random variable representing the random amount of money paid by an insurance company to indemnify a policyholder, a beneficiary and/or a third-party in the execution of an insurance contract.

The following We state the definition of a risk measure presented by Denuit ((Denuit et al., 2006)).

Definition 3.2. A risk measure is a functional $\wp$ mapping a risk R to a non-negative real number $\wp(R)$, possibly infinite, representing the extra cash that has to be added to R to make it acceptable.

The idea is that $\wp$ quantifies the riskiness of R: large values of $\wp(R)$ tell us that R is dangerous. Specifically, if R is a possible loss of some financial portfolio over a time horizon, we interpret $\wp(R)$ as the amount of capital that should be added as a buffer to this portfolio so that it becomes acceptable to an internal or external risk controller. In such a case, $\wp(R)$ is the risk capital of the portfolio. Such risk measures are used for determining provisions and capital requirements in order to avoid insolvency; see Panjer (Panjer et al., 1998).

3.2. DESIRABLE PROPERTIES

The following conditions are of interest for a risk measure:

1. $\wp(X) \leq \max(X) = F_X^{-1}(1)$ (Non-excessive loading)
2. $\wp(X) \leq E(X)$ (Non-negative loading)
3. $\wp(X+c) = \wp(X) + c$ for each constant c(Translativity)
4. $\wp(c) = c$(Constancy)
5. $\wp(X+Y) \leq \wp(X) + \wp(Y)$ for all random variables X and Y(Subadditivity)
6. $\wp(X+Y) = \wp(X) + \wp(Y)$ for all comonotonic random variables X and Y (Comonotonic additivity)
7. $\wp(cX) = c\wp(X)$(Positive homogeneity)
8. $Pr(X \leq Y) = 1 \Rightarrow \wp(X) \leq \wp(Y)$ (Monotonicity)
9. Let $\{X_n, n = 1, 2, ...\}$ be a sequence of risks such that $X_n \to X$ as $n \to \infty$, that is,
$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{3.1}$$
for every continuity point x of F_X. Then,
$$\lim_{n\to\infty} \wp_{X_n}(x) = \wp_X(x) \tag{3.2}$$
(Continuity with respect to convergence in distribution)

Several authors have chosen some of these conditions to form a set of requirements that any risk measure must meet. A risk measure is called a coherent risk measure if it satisfies axioms (3) translative, (5) subadditive, (7) positive homogeneous and (8)monotone is called coherent. We can use the Lorenz curve as a measure of risk. The Lorenz curve shows the cumulative share of loss or gain from different sections of the portfolio. $\forall c > 0, \quad \alpha \in [0,1]$

$$\begin{aligned} L_{cX}(\alpha) &= \frac{1}{c\mu}\int_0^\alpha F_{cX}^{-1}(t)dt \\ &= \frac{1}{c\mu}\int_0^\alpha cF_X^{-1}(t)dt \\ &= L_X(\alpha) \end{aligned} \tag{3.3}$$

The Lorenz curve is positive homogeneity or in other words, it is invariant with positive scaling. This characteristic indicates that inequality in society does not depend on the scale of measurement. $\forall c > 0, \quad \alpha \in [0,1]$

$$\begin{aligned} L_{X+c}(\alpha) &= \frac{1}{\mu+c}\int_0^{\alpha} F_{X+c}^{-1}(t)dt \\ &= \frac{1}{\mu+c}\int_0^{\alpha} (F_X^{-1}+c)(t)dt \\ &= \frac{\mu}{\mu+c}L_X(\alpha) + \frac{c}{\mu+c}\alpha \end{aligned} \tag{3.4}$$

therefore

$$\begin{aligned} \mu L_{X+c}(\alpha) + cL_{X+c}(\alpha) &= \mu L_X(\alpha) + c\alpha \\ \mu(L_{X+c}(\alpha) - L_X(\alpha)) &= c(\alpha - L_X(\alpha)) \end{aligned} \tag{3.5}$$

Since we have $L(p) \leq p$ for each $p \in [0,1]$ so $L_{X+c}(\alpha) \geq L_X(\alpha))$. This means that inequality is reduced by adding a fixed amount to the income of people in society Since $X \leq \max(X)$ we have that $F^{-1}(p) \leq \max(X)$ so that $L(p) \leq \frac{p\max(X)}{\mu}$. It can easily be shown that the Lorenz curve does not have any of the properties of 4 and 6 either. For any $p > 0, \quad F^{-1}(p) = c$ we have $L(p) = \frac{cp}{\mu}$. And similar to the previous cases, it can be shown that it has the features of 5, 8 and .

- $L(p+q) \leq L(p) + L(q)$ for all p and q in $[0,1]$(Subadditivity)
- $Pr(X \leq Y) = 1 \Rightarrow L_X(p) \leq L_Y(p)$ (Monotonicity)
- Let $\{X_n, n = 1,2,...\}$ be a sequence of risks such that $X_n \to X$ as $n \to \infty$, that is,

  $$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{3.6}$$

  for every continuity point x of F_X. Then,

  $$\lim_{n\to\infty} L_{X_n}(p) = L_X(p) \tag{3.7}$$

  (Continuity with respect to convergence in distribution)

Therefore, the Lorenz curve is not a coherent risk.

3.3. VALUE-AT-RISK

Value at risk (VaR) is a statistical technique used to measure and determine the amount of financial risk in a company or investment portfolio over a period of time. In general, it can be said that the value at risk measures the maximum amount of expected loss in a certain time horizon at a certain level of confidence and is measured by 3 variables: the amount of potential loss, the probability of potential loss and the time period. VaR is defined as follows.

Definition 3.3. For a fixed probability level α, the value-at-risk denoted by VaR_X is defined as

$$VaR_X(\alpha) = F_X^{-1}(\alpha). \tag{3.8}$$

Equation (3.8) for all $x \in R$ and $p \in (0,1)$ is equivalent to the following equation.

$$VaR_X(p) \leq x \Leftrightarrow p \leq F_X(x). \tag{3.9}$$

You can see properties of this measure in the (Denuit et al., 2006). VaR can have several equivalent interpretations((Kisiala, 2015)) :

- $VaR_X(\alpha)$ is the minimum loss that will not be exceeded with probability α.
- $VaR_X(\alpha)$ is the α-quantile of the distribution of X.
- $VaR_X(\alpha)$ is the smallest loss in the $(1-\alpha)\times 100\%$ worst cases.
- $VaR_X(\alpha)$ is the highest loss in the $\alpha \times 100\%$ best cases.

It can be shown that the value at risk has the following properties:

1. $VaR_X(\alpha) \leq \max(X)$.
2. $\forall \alpha > 0, VaR_c(\alpha) = c$

The value at risk is non-excessive loading, translative, comonotonic additive, monotone, positively homogeneous and is continuous with respect to convergence in distribution. VaR is an optimal capital requirement for insurance companies. The value at risk does not necessarily include entail non-negative loading, sub-additive. There are several reasons explaining the success of CVaR as a risk measure. From a practical point of view, CVaR penalizes only the negative deviations with respect to an efficiency target (downside risk measure); it is sensitive to extremely negative outcomes but it is not as conservative as a Minimax approach. From a theoretical viewpoint, CVaR is a coherent risk measure, i.e., it guarantees the consistency with intuitions about rational risk-averse decision makers. From a computational point of view, CVaR optimization can often be embedded in an optimization problem by adding linear constraints and continuous variables, i.e., without increasing the expected complexity of the resulting optimization model. Therefore, it is not surprising that in recent years several authors incorporate CVaR as an additional criterion in their optimization problems, even while facing a number of applications different from optimization in finance ((Filippi et al., 2017)). CVaR is strongly related to another risk measure called Value-at-Risk, or VaR for short, which is heavily used in various financial and engineering problems, including military, nuclear, and airspace applications ((Sarykalin et al., 2008)). Intuitively speaking, the VaR of a portfolio of assets, given a specified probability level α, can be defined as the smallest threshold value η such that the probability that the loss on the portfolio exceeds η is α. The value α is chosen by the decision maker, and is often called confidence level. As pointed out by Rockafellar and Uryasev (Rockafellar and Uryasev, 2002), a very serious shortcoming of VaR is that it does not provide any indication about the severity of losses beyond its value. Indeed, Sarykalin et al. (Sarykalin et al., 2008) highlight that one can significantly increase the largest loss exceeding the VaR, but the VaR risk measure will not change. The CVaR overcomes this limit affecting the VaR, as it measures the conditional expectation of losses above η (see (Rockafellar et al., 2000)) (Filippi et al., 2017) survey reviews 88 papers, all dating from 2005 or later, where the concept of CVaR is embedded into a decision problem arising in a non-financial context. The related literature is growing ceaselessly. The classification of the papers reviewed in this survey is depicted in Figure 2 Filippi et al. (Filippi et al., 2017) presented those articles that deal with a classical topic in the Operations Research and Management Science (OR/MS) literature. These topics are Inventory management, Supply chain management, Transportation and traffic control, Location and supply chain network design, Networks, and Scheduling. Altogether they represent roughly 70% of the articles surveyed. A different topic is Energy, which covers issues specifically related to the supply chain in the energy sector. Medicine concerns a specific application of radiation therapy, where CVaR is applied in a deterministic context. Finally, the label Other is referred to a number of different applications, that do not fit in any the above categories.

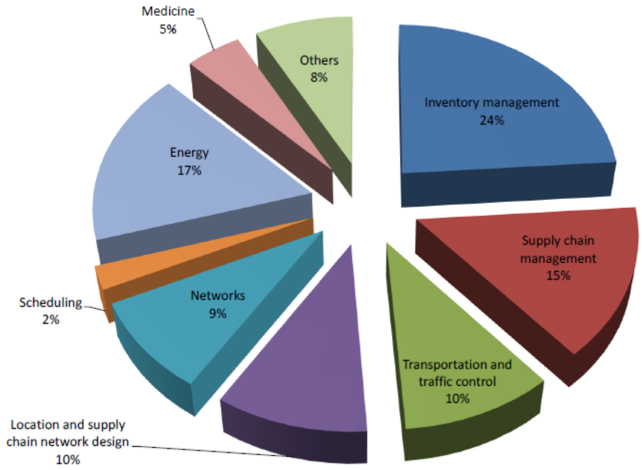

Figure 1: **The classification per area of the literature reviewed ((Filippi et al., 2017)).**

3.4. TAIL VALUE-AT-RISK

A VaR at the preset level α does not distribute any information about the thickness of the high-performance sequence. This is a significant drawback because in practice the regulator is related not only to the default frequency but also to the default intensity. Shareholders and management should also be concerned about the question, "How bad is it?" When they want to assess the risks in a consistent way. Therefore, another frequently used risk criterion is called the tail value at risk (TVaR) and is defined below.

Definition 3.4. The tail value at risk TVaR, denoted by $TVaR_X(p)$ for a risk X and a probability level p, is defined as

$$TVaR_X(p) = \frac{1}{1-p}\int_p^1 VaR_X(t)dt, \quad 0 < p < 1. \tag{3.10}$$

Hence, $TVaR_X(p)$ also can be interpreted as the arithmetic average of the VaRs of X, from p on.

Note that when $p = 0$, then $TVaR_X(p)$ is equal to the mean of X. TVaR is no-ripoff, TVaR does not induce unjustified loading, induces a non-negative loading, is translative, positively homogeneous, comonotonic additive and monotone, is subadditive.

Alternative names for TVaR in the sources include: Average Value-at-Risk, Expected Shortfall, Conditional Value-at-Risk or Tail Conditional Expectation, although some authors make subtle differences between their definitions((Kisiala, 2015)). The conditional value-at-risk $CVaR_X$ is defined as the conditional expectation of X, given that $X \geq VaR_X$, i.e.

$$CVaR_X = E(X|X > VaR_X(\alpha)). \tag{3.11}$$

Figure 2 shows the value at risk and conditional value-at-risk for a continuous random variable X. The $VaR_X(\alpha)$ can be calculated using the cumulative distribution function of X and $CVaR_X(\alpha)$ can be calculated from $VaR_X(\alpha)$.

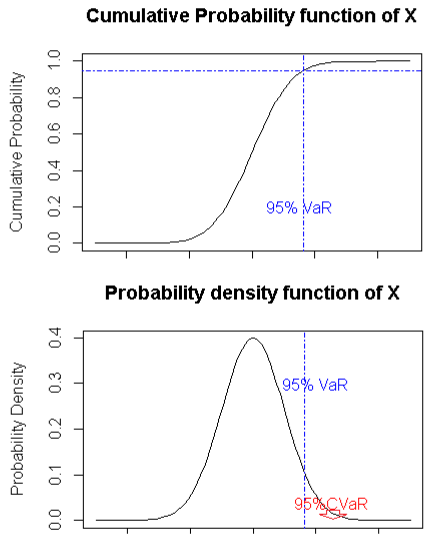

Figure 2: VaR_X **and** $CVaR_X$ **of the normal random variable expressing damage.**

Example 3.5. For U from estandard uniform(Denuit et al., 2006),

$$E(X) = E(F_X^{-1}(U)) = \int_0^1 F_X^{-1}(t)dt = TVaR_X(0), \tag{3.12}$$

3.5. WANG RISK MEASURES

The Wang risk measure is a class of risk measures. This class encompasses several indicators such as the Value-at-Risk and the Conditional Value-at-Risk. Can be shown the aforesaid quantities can be written as simple combinations of

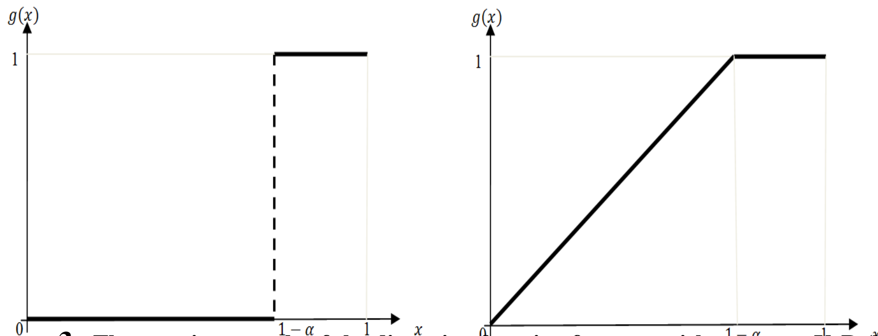

Figure 3: **The certain example of the distortion function for wang risk measure, VaR (left) and CVaR(right).**

Wang distortion risk measures. Wang risk measure is weighted average of the quantile function. The Wang risk measure can be useful to price insurance premiums, bonds, and tackle capital allocation problems((Wang, 2004)). The Wang risk measure is defined next.

Definition 3.6. Let X is a non-negative random variable. The Wang risk measure ρ_g of risk X is defined as

$$\rho_g(X) = \int_0^\infty g(1 - F_X(x))dx. \tag{3.13}$$

for some right-continuous and non-decreasing, with $g(0) = 0$ and $g(1) = 1$. A function with these properties is called a distortion function.

$\rho_g(X)$ can be written as

$$\rho_g(X) = \int_0^1 g(t)dF^{-1}(1-t) = \int_0^1 F^{-1}(1-t)dg(t). \tag{3.14}$$

A Wang risk maesure is a weighted version of the expectation of the random variable X. Easily seen certain examples contain

- If $I\{\cdot\}$ be the indicator function and $g(x) = I\{x \geq 1 - \alpha\}$ then $\rho_g(X)$ is equal to $VaR_X(\alpha)$.
- If $g(x) = \min(\frac{x}{1-\alpha}, 1)$, then $\rho_g(X)$ is equal to $TVaR_X(\alpha)$.

We know that the Lorenz curve is convex and $L(0) = 0, \quad L(1) = 1$, so if $g(x) = 1 - L(1-x)$ then $g(x)$ is a distortion function and the wang risk measure is equal to:

$$\rho_g(X) = \int_0^\infty (1 - L(F_X(x)))dx = \int_0^1 \overline{L}(1-t)dF^{-1}(1-t). \tag{3.15}$$

Results in $\rho_g(X) = \dfrac{G-1}{2} + F^{-1}(1) - F^{-1}(0)$ Which G is the Gini coefficient. The stationary renewal distribution is an important concept in ruin theory. Let us recall the definition of the stationary renewal distribution.

Definition 3.7. For a non-negative random variable X with finite mean, let $X_{[1]}$ denote an random variable with distribution function

$$F_{X_{[1]}}(x) = \frac{1}{E(X)} \int_0^x \overline{F}_X(t)dt = 1 - \frac{EW(x)}{E(X)}, \quad x \geq 0. \tag{3.16}$$

The distribution function $F_{X_{[1]}}$ is known as the stationary renewal distribution associated with X.

Example 3.8. Let $X \sim N(\mu, \sigma^2)$. Then, we have $\alpha \in (0,1)$

$$VaR_X(\alpha) = \mu + \sigma\Phi^{-1}(\alpha), \tag{3.17}$$

$$EW_X(x) = \sigma\Phi'\left(\frac{x-\mu}{\sigma}\right) - (x-\mu)\left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right) \tag{3.18}$$

$$TVar(\alpha) = \mu + \sigma\frac{\Phi'(\Phi^{-1}(\alpha))}{1-\alpha} \tag{3.19}$$

$$ES = \sigma\Phi'(\Phi^{-1}(\alpha)) - \sigma\Phi^{-1}(\alpha)(1-\alpha) \tag{3.20}$$

where Φ denotes the standard normal distribution function.

Example 3.9. Let $X \sim LN(\mu, \sigma^2)$. Then, we have $\alpha \in (0,1)$

$$VaR_X(\alpha) = \exp(\mu + \sigma\Phi^{-1}(\alpha)), \alpha \in (0,1) \tag{3.21}$$

$$EW_X(x) = \exp(\mu + \frac{\sigma^2}{2})\Phi(x_1) - x\Phi(x_2), \quad x > 0 \tag{3.22}$$

$$TVar(\alpha) = \exp(\mu + \frac{\sigma^2}{2})\frac{\Phi(\sigma - \Phi^{-1}(\alpha))}{1-\alpha} \tag{3.23}$$

$$ES = \exp(\mu + \frac{\sigma^2}{2})\Phi(\sigma - \Phi^{-1}(\alpha)) - \exp\left(\mu + \sigma\Phi^{-1}(\alpha)\right)(1-\alpha) \tag{3.24}$$

where Φ denotes the standard normal distribution function, $x_1 = \dfrac{\mu - \ln x}{\sigma} + \sigma$ and $x_2 = x_1 - \sigma$.

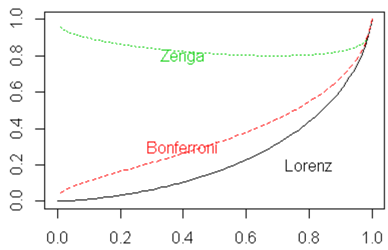

Figure 4: **Income inequality indices of the Lognormal distribution.**

4. RELIABILITY CONCEPTS

Lifetime variables and data are one of the most important variables and data in the world today, which are studied in detail in a branch of statistics called reliability. Lifetime variables and data are one of the most important variables and data in the world today, which are studied in detail in a branch of statistics called reliability.

Reliability is one of the most important quality characteristics of large and complex components, products and systems, which has a significant role and importance in evaluating the objectives and examining their current status. Nowadays, the necessity of discussing reliability and similar topics in all practical aspects has been accepted as an indisputable principle and need. The issue of reliability arose in the nineteenth century to help the marine and life insurance industry to more accurately calculate the amount they received from the customer. Reliability plays an essential and undeniable role in industry, technology, medicine and other sciences. There are different definitions of reliability in statistical texts. In fact, it can be defined as the probability of a system performing satisfactorily at a given time and under certain operating conditions. Statistically, reliability is the compatibility of a set of dimensions or measuring instruments often used to describe an experiment. In following we give a brief note of the basic concepts and results in reliability theory, which are used in the sequel and are referred in the text. The commonly used concepts in reliability theory are:
(i) the survival function, (ii) the failure rate, (iii) the mean residual life function. The survival function or tail function of F is signified by $\overline{F} = 1 - F$ The hazard

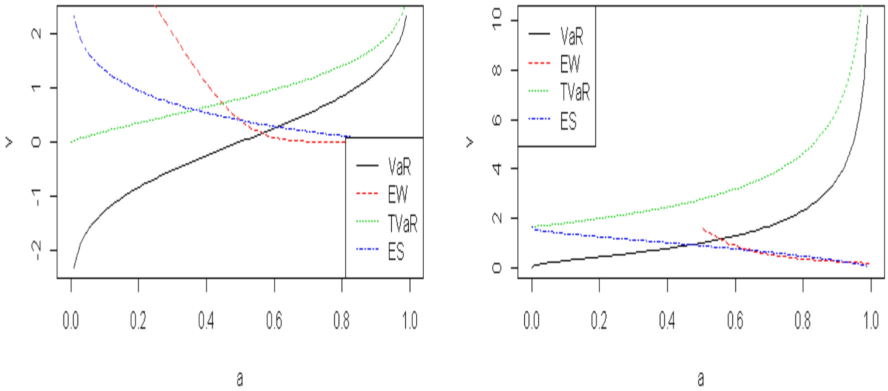

Figure 5: **The Risk measures for Normal (left) and (right) for Lognormal.**

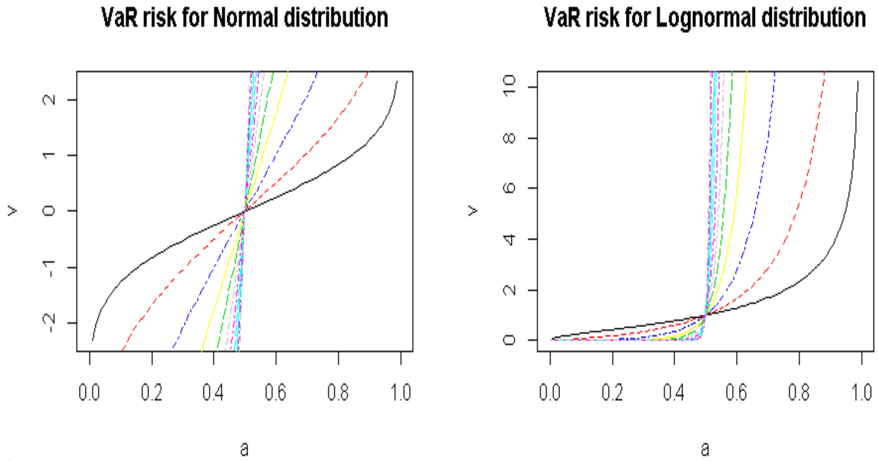

Figure 6: **The VaR for Normal (left) and Lognormal (right) for different variances(ranging from one to six from right to left).**

rate or failure rate function $r_X(t)$ is defined as $r_X(t) = \dfrac{f_X(t)}{\overline{F}_X(t)}$ for $t \geq 0$. The hazard rate corresponds to the intensity of mortality in life insurance. Let X be a non-negative random lifetime with $cdf\ F(.)$ with a finite moment. The residual holding in excess of l given $X > l$ represents $X - l|X > l$. The mean residual life (MRL) is defined as:

$$m_F(x) = E(X - t|X > t) = \frac{\int_x^\infty \overline{F}(t)dt}{\overline{F}(x)}, \quad x \geq 0 \tag{4.1}$$

If $F(.)$ is absolutely continuous, then MRL can be rewritten as

$$m_F(x) = \frac{\int_x^\infty tf(t)dt}{\overline{F}(x)} - x = \nu(x) - x, \quad x > 0. \tag{4.2}$$

In economics, $m_F(x)$ describes the extent of wealth the proportion of rich beyond t in the population command. The function $\nu(t) = E(X|X > t) = \dfrac{\int_t^\infty xf(x)dx}{\overline{F}(t)}$ is known as the vitality function (VF) or life expectancy. The VF and MRL play important roles in engineering reliability, biomedical science, and survival analyzes. Comparing VF with Equation (3.11), it is clear that these two are a formula that have different applications in reliability and insurance.

5. STOCHASTIC ORDER

Stochastic orderings are tools used in reliability, insurance, finance, and other fields to compare characteristics of interest, such as location, variability or shape, of probability distributions. Stochastic orders have shown to be useful notions in several areas of economics, the inequality analysis, risk analysis, reliability or portfolio insurance. Since the 1970, stochastic dominance rules have been used in comparison and analysis of poverty and income inequality.

Definition 5.1. Let X and Y be two random variables with distribution functions F and G, respectively. Then,X is said to be smaller than Y:

(a) In the usual stochstic order (denoted by $X \leq_{st} Y$) if $F^{-1}(p) \leq G^{-1}(p)$, far all $p \in (0,1)$.

(b) In the dispersive order (denoted by $X \leq_{disp} Y$) if $F^{-1}(p) - F^{-1}(q) \leq G^{-1}(p) - G^{-1}(q)$, for all $0 < q < p < 1$.

(c) Stoehastie dominance of order 1(denoted by $X \leq_{SD(1)} Y$) iff $E[\phi(X)] \leq E[\phi(Y)]$ for all integrable monotonic function ϕ.

(d) Stoehastie dominance of order 2(denoted by $X \leq_{SD(2)} Y$) iff $E[\phi(X)] \leq E[\phi(Y)]$ for all integrable concave, monotonic function ϕ.

(e) Monotonic dominance of order 1(denoted by $X \leq_{MD(1)} Y$) iff $E[\phi(X)] \leq E[\phi(Y)]$ for all integrable concave function ϕ.

Definition 5.2. The random variable X is smaller than Y (over the union of the supports of X and Y) in the:

- Convex order ($X \leq_c Y$), if $F_Y^{-1}(F_X(x))$ is convex on the support of X.
- Star shaped order ($X \leq_* Y$), if $\dfrac{F_Y^{-1}(t)}{F_X^{-1}(t)}$ is an increasing function in $t \in (0,1)$.
- Lorenz order ($X \leq_L Y$) if $L_Y(u) \leq L_X(u)$ for all $0 \leq u \leq 1$.

It should be mentioned that

$$X \leq_c Y \Longrightarrow X \leq_* Y \Longrightarrow X \leq_L Y \Longrightarrow Y \leq_G X. \tag{5.1}$$

6. MAIN RESULTS

6.1. RELATIONSHIPS BETWEEN RISK MEASURES AND LORENZ CURVE

Following, we state the properties of Lorenz curve and $CVaR_X$. Part of this relations is expressed in reference (Pflug, 2000) and other parts can easily be proved.

Remark 6.1. Let F_X is continuous. For any $p \in (0,1)$, the following identities are valid:

- $TVaR_X(p) = VaR_X(p) + \dfrac{1}{1-p} EW_X(p)$.
- $CVaR_X(p) = \dfrac{EW_X(p)}{\overline{F}_X(VaR_X(p))}$.
- $CVaR_X, VaR_X$ (Lorenz curve) is (is not) translation-equivariant.
- $CVaR_X, VaR_X$ (Lorenz curve) is (is not) positively homogeneous.
- $CVaR_X$ (Lorenz curve) is (is) convex.

- If Y has a density, $E(Y) = (1-\alpha)CVaR_X(Y) - \alpha CVaR_{1-\alpha}(-Y)$, $VaR_X(Y) = -VaR_{(1-\alpha)}(-Y)$ $(E(Y) = F^{-1}(p), iff \quad L'(p) = 1)$.
- $CVaR_X$ is monotonie w.r.t. SD(2) and MD(2).
- VaR_X is comonotone additive and is monotonic w.r.t. SD(1).

In principle, VaR and CVaR measure different properties of the distribution. VaR is a quantile and CVaR is a conditional tail expectation. The two values coincide only? if the tail is cut off. Let $[Y]^c$ be the right censored cost variable ($[Y]^c = min(Y,c)$). If we set $c = VaR_X$ then $CVaR_X([Y]^c) = VaR_X(Y)$(Pflug (2000)).

Proposition 6.2. *The following statement is taken from (Pflug, 2000):*

- $CVaR_X(Y) \geq VaR_X(Y)$.
- $VaR_X(Y) = \sup\{v : CVaR_X([Y]^v) = v\}$.
- *If* Y *is nonnegative, then* $\left[\dfrac{E(Y^n) - (1-\alpha)CVaR_X(Y^n)}{\alpha}\right]^{\frac{1}{n}} \to VaR_X(Y)$ *as* $n \to \infty$.

In here, some of the important results about the relationship between risk measures and some measures of income inequality are presented.

Proposition 6.3. *It is easy to see that, when the distribution function is continuous,*

$$VaR_X(\alpha) = \mu L'(\alpha) \tag{6.1}$$

$$L''(\alpha) f(VaR_X(\alpha)) = \frac{1}{\mu}, \tag{6.2}$$

also, we have the following equalities for all $\alpha \in (0,1)$*:*

$$CVaR_X(\alpha) = \frac{\mu(1-L(\alpha))}{1-\alpha} \tag{6.3}$$

$$CVaR_X(\alpha) = \frac{\mu(1-\alpha B(\alpha))}{1-\alpha} \tag{6.4}$$

$$CVaR_X(\alpha) = \frac{\mu}{1-\alpha Z(\alpha)} \tag{6.5}$$

$$\tag{6.6}$$

And

$$m_F(VaR_X(\alpha)) = \mu \frac{\overline{L}(\alpha)}{1-\alpha} - x, \tag{6.7}$$

$$m_F(VaR_X(\alpha)) = CVaR_X(\alpha) - x. \tag{6.8}$$

where $\overline{L}(u) = 1 - L(u)$.

Table 1: Some distributions and their properties.

Name	$F(x)$	$L(x)$	$CVaR_x$
Exponential	$1-e^{-\lambda x}; x \geq 0$	$x+(1-x)\log(1-x)$	$\frac{1-\log(1-x)}{\lambda}$
Uniform	$\frac{x-a}{\theta}; a \leq x \leq a+\theta$	$\frac{2ax-\theta x^2}{2a+\theta}$	$\frac{\theta}{2}\,\frac{2a+\theta(1+x)}{2a+\theta}$
Weibull	$1-\exp\{-(\lambda x)\}^{\alpha}; \alpha, \lambda > 0$	$1-\frac{\Gamma(-\log(1-x), 1+1/\alpha)}{\Gamma(1+1/\alpha)}$	$\frac{\lambda\Gamma(-\log(1-x), 1+1/\alpha)}{1-x}$
Lognormal	$\Phi\left\{\frac{\log x-\alpha}{\sigma}\right\}; \sigma > 0, x \geq 0$	$\Phi(\Phi^{-1}(x)-\sigma)$	$\frac{\exp(\mu+\sigma^2/2)(1-\Phi(\Phi^{-1}(x)-\sigma))}{1-x}$
Pareto	$1-(\frac{x}{\beta})^{-\alpha}; x \geq \beta > 0$	$1-(1-x)^{\frac{\alpha-1}{\alpha}}\ \alpha > 1$	$\frac{\alpha^3\beta^2}{(\alpha-1)(\alpha-2)}(1-x)^{\frac{1}{\alpha}}$
Exponential	$1-e^{\{-(\lambda x)\}}; \lambda > 0$	$x+(1-x)\log(1-x)$	$\frac{1+\log(1-x)}{\lambda}$

The following proposition presents stochastic orders in terms risk measures.

Proposition 6.4. *The random variable X is smaller than Y (over the union of the supports of X and Y) in the:*

- *Convex order ($X \leq_c Y$), if $VaR_{(F_X(x))}(Y)$ is convex on the support of X.*
- *Star shaped order ($X \leq_* Y$), if $\frac{VaR_t(Y)}{VaR_t(X)}$ is an increasing function in $t \in (0,1)$.*
- *Lorenz order ($X \leq_L Y$) if $\frac{CVaR_u(X)}{E(X)} \leq \frac{CVaR_u(Y)}{E(Y)}$ for all $0 \leq u \leq 1$.*

Other characterizations of interest are the following. A lower (upper) bound for Lorenz curve is obtained when F is IFR (DFR), this matter is explained by the following theorem:

Theorem 6.5. *Let X be non-negative and continuous random variable with distribution function F and Lorenz curve L. If X is IFR (DFR) then $\mu L(p) \geq (\leq)VaR_p(p-1) - \frac{pVaR_p}{\ln(1-p)}$.*

Proof. Barlow and Proschan (Barlow and Proschan (1996)) showed that if F is IFR (DFR) then

$$\begin{cases} \overline{F}(t) \geq (\leq) e^{-at} & ; \quad t \leq F^{-1}(p), \\ \overline{F}(t) \leq (\geq) e^{-at} & ; \quad t \geq F^{-1}(p), \end{cases} \tag{6.9}$$

where $a = -\dfrac{\ln(1-p)}{F^{-1}(p)}$. Considering the definition of the Lorenz curve we have

$$\begin{aligned} \mu L(p) = \int_0^{F^{-1}(p)} t f(t) dt &= F^{-1}(p)(p-1) + \int_0^{F^{-1}(p)} \overline{F}(t) dt, \\ &\geq (\leq) F^{-1}(p)(p-1) + \int_0^{F^{-1}(p)} e^{-at} dt, \\ &= F^{-1}(p)(p-1) + \frac{F^{-1}(p)(\exp(\ln(1-p)-1))}{\ln(1-p)}, \\ &= F^{-1}(p)(p-1) - \frac{pF^{-1}(p)}{\ln(1-p)}, \end{aligned} \tag{6.10}$$

and using Equation (3.21) the proof is complete. □

Remark 6.6. It is worthwhile to note that if $VaR_p = -\dfrac{\ln(1-p)}{a}$, it can be concluded $L(p) \geq (\leq) \dfrac{1}{a}[(1-p)\ln(1-p)+p]$ where $(1-p)\ln(1-p)+p$ is exponential Lorenz curve in Theorem 6.5, it shows the Lorenz curves of distributions, which have increasing failure rate, have the Lorenz curve of an exponential distribution as lower bounded.

Theorem 6.7. *Suppose X_1 and X_2 be two non-negative random variables with finite means. Let VaR_1 and VaR_2 are the value at risk for X_1 and X_2 respectively. $W(t) = \dfrac{VaR_1(t)}{VaR_2(t)}$ for all $t \in (0,1)$ we have:*

- *If $W(t)$ be increasing then $X_2 \leq_L X_1$.*
- *If $W(t)$ be decreasing then $X_1 \leq_L X_2$.*
- *If $W(t)$ be constant then $X_2 =_L X_1$.*

Proof. To get the result, let us use Definition Lorenz order, which gives

$$\begin{aligned} X_1 \leq_L X_2 \quad &\Leftrightarrow L_1(\alpha) \geq L_2(\alpha)\, for\, all\, \alpha \in (0,1) \\ &\Leftrightarrow L_1(\alpha) - L_2(\alpha) \geq 0. \end{aligned} \tag{6.11}$$

Suffice it to show that $\int_0^\alpha \mu_1^{-1} VaR_1(t) - \mu_2^{-1} VaR_2(t)$ is a non-negative function. Let us define

$$D(t) = \qquad L_1(t) - L_2(t) \tag{6.12}$$

$$= \quad \frac{\mu_2}{\mu_1}\left(\int_0^\alpha \left(\frac{VaR_1(t)}{VaR_2(t)} - \frac{\mu_2}{\mu_1}\right)\right) \tag{6.13}$$

Note that $L_1(0) = L_2(0) = 0$ and $L_1(1) = L_2(1) = 1$, so $D(0) = D(1) = 0$. The result is obtained according to the continuity of the function and the use of the Roll's theorem. □

Example 6.8. Let $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{2}$ and $P(X_2 = 2) = P(X_2 = 5) = \frac{1}{2}$. We calculate

$$\frac{VaR_2(\alpha)}{VaR_1(\alpha)} \begin{cases} 2 & ; \quad 0 < \alpha \leq \frac{1}{2}, \\ \frac{5}{2} & ; \frac{1}{2} < \alpha < 1, \end{cases} \tag{6.14}$$

Since $\frac{VaR_2(\alpha)}{VaR_1(\alpha)}$ is an increasing function with α, therefor $X_1 \leq_L X_2$.

Property 6.9. Let X be an random variable For any $0 < p < 1$, the following equalities hold (Denuit et al., 2006)

- If t is non-decreasing and continuous then $F_{t(X)}^{-1}(p) = t(F_X^{-1}(p))$.
- If t is non-decreasing and continuous then $F_{t(X)}^{-1+}(p) = t(F_X^{-1+}(p))$.
- $F_X^{-1}(p) = \overline{F}_X^{-1}(1-p)$ and $F_X^{-1+}(p) = \overline{F}_X^{-1+}(1-p)$

Example 6.10. In this example, in order to use real data, the numbers related to the S&P 500 index (daily close) during the two periods 2020-6-1-2022-6-1 and 2018-6-1-2020-6-1 have been used. Calculations related to data analysis, and graphs are performed using R and EXCEL software. Estimation of Lorenz curve ,that has not to do with income distribution but with the S&P 500 index data, and VaR is performed by empirical method. Figure 7 presentations the daily chart of S&P 500 index in the days of June 2018 to June 1, 2020 and similarly in the years 2020-2022. As shown in the figure, the lowest value of the index is related to March 23, 2020 and the highest value of the index is related to June 9, 2020. Figure 8 shows the Lorenz curve on the left and the VaR risk curve on the right for the the S&P 500 index during the two periods 2020-6-1-2022-6-1 and 2018-6-1-2020-6-1. Using the proposition 6.4, if $\dfrac{VaR_X}{VaR_Y}$ $(X = 2020-2022$ and

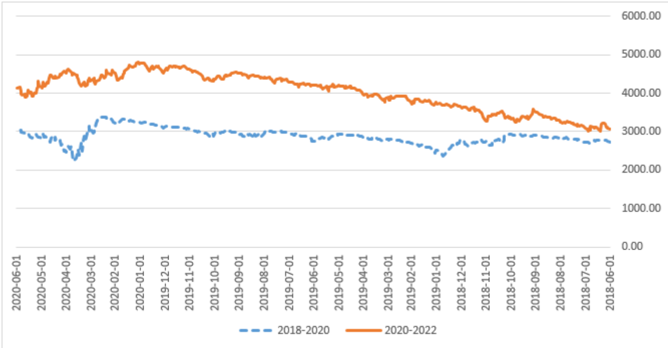

Figure 7: **S&P 500 daily index chart for the first days of June 2018 to the first of June 2020 and similarly for the years 2020-2022.**

$Y = 2018-2020)$ is increasing, then the star order is established between these variables, and therefore according to Figure 9, it can be concluded that there is no star order between the data in these two years, and as a result, the Lorenz order will not be established, which is quite clear in Figure 8. Theorem6.5, on the other hand, clearly defines the Lorenz relationship between data using their VaR risk function. According to this case, in the period when the$W(t) = \dfrac{VaR_X}{VaR_Y}$ $(X = 2020-2022$ and $Y = 2018-2020)$ is increasing, 2018-2020 has less inequality than in 2020-2022, and in the period when the $W(t)$ is ascending, 2018-2020 will have more inequality than in 2020-2022. The results of this theorem are clearly shown in Figures 8, 9 and Table 2.

7. CONCLUSION

Many important life decisions need to be considered and lightened and weighted forward. Having financial knowledge and recognizing risk or return can help you evaluate decision options. The Lorenz curve is an old measure in the economics and the other branches of science to evaluate the income inequality of society. In this paper we have obtained important and new properties of risk measures such

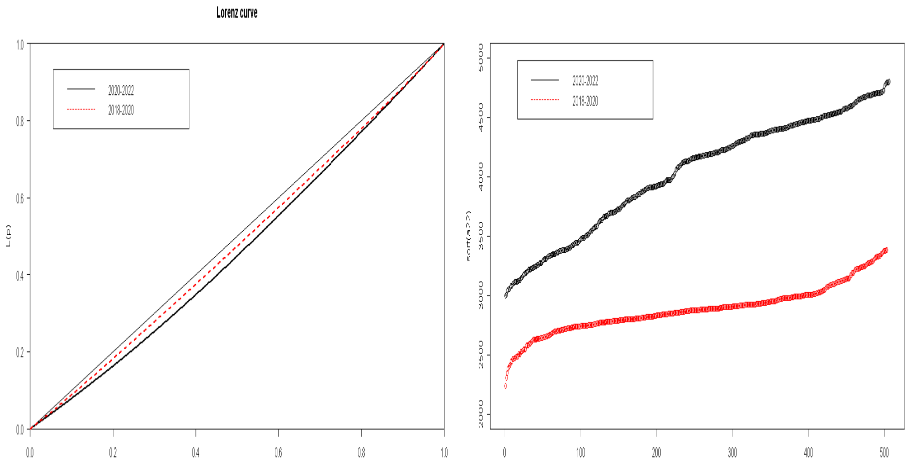

Figure 8: The Lorenz curve (left) and Value at risk(right) for S&P 500 index during the two periods 2020-2022 and 2018-2020.

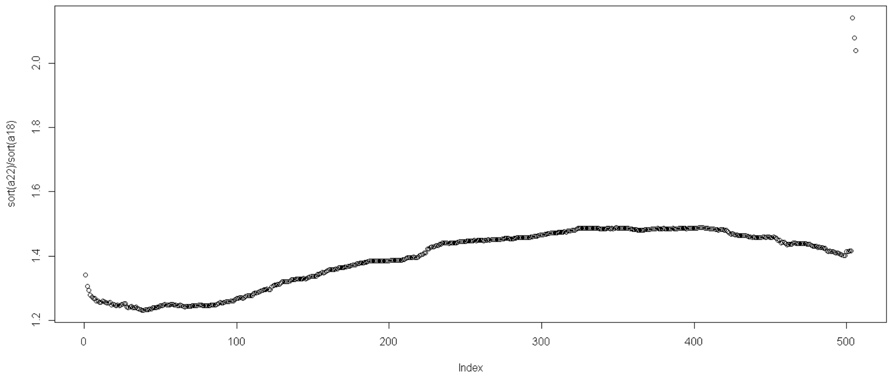

Figure 9: Graphs $\frac{VaR_X}{VaR_Y}$ for S&P 500 index data.($X = 2020 - 2022$ and $Y = 2018 - 2020$).

as value at risk and conditional value at risk. We have derived simpler expressions for many relevant economic inequality and risk indices using reliability contects. It was, also to concentrate on some properties of aging classes based on the Lorenz curve and the other measures of risk. This paper indicates that the Lorenz curve and risk measures can be used to qualify the aging concept of lifetime distribu-

Table 2: The values of the Lorenz curve and the VaR risk function are specified in some places ($X = 2020 - 2022$ and $Y = 2018 - 2020$).

p	$L_X(p)$	$L_Y(p)$	$VaR_X(p)$	$VaR_Y(p)$
0.125	0.1	0.11	3349.425	2688.705
0.25	0.21	0.23	3636.145	2771.90
0.375	0.32	0.35	3907.764	2820.10
0.5	0.45	0.47	4165.555	2878.38
0.625	0.58	0.6	4315.290	2919.392
0.75	0.72	0.73	4435.638	2984.645
0.825	0.80	0.81	4500.410	3030.69

tions. The following conclusions were drawn from this research. The bound of the class of lifetime distributions are determined in terms of its Lorenz curve and the other index inequalities. Some interesting relationships that exist between commonly used notions in economic theory and reliability. Some new properties for income inequality are obtained.

References

Arnold, B.C. (2012). *Majorization and the Lorenz order: A brief introduction*, vol. 43. Springer Science & Business Media.

Arnold, B.C. (2015). On zenga and bonferroni curves. In *Metron*, 73 (1): 25–30.

Barlow, R.E. and Proschan, F. (1996). *Mathematical theory of reliability*. SIAM.

Behdani, Z., Borzadaran, G.R.M., and Gildeh, B.S. (2018). Relationship between the weighted distributions and some inequality measures. In *Communications in Statistics-Theory and Methods*, 47 (22): 5573–5589.

Behdani, Z., Borzadaran, G.R.M., and Gildeh, B.S. (2019). Connection of generalized failure rate and generalized reversed failure rate with inequality curves. In *International Journal of Reliability, Quality and Safety Engineering*, 26 (02): 1950006.

Behdani, Z., Mohtashami Borzadaran, G.R., and Sadeghpour Gildeh, B. (2020). Some properties of double truncated distributions and their application in view of income inequality. In *Computational Statistics*, 35 (1): 359–378.

Belzunce, F., Riquelme, C.M., and Mulero, J. (2015). *An introduction to stochastic orders*. Academic Press.

Berkhouch, M., Lakhnati, G., and Righi, M.B. (2018). Extended gini-type measures of risk and variability. In *Applied Mathematical Finance*, 25 (3): 295–314.

Berkhouch, M., Müller, F.M., Lakhnati, G., and Righi, M.B. (2022). Deviation-based model risk measures. In *Computational Economics*, 59 (2): 527–547.

Chandra, M. and Singpurwalla, N.D. (1981). Relationships between some notions which are common to reliability theory and economics. In *Mathematics of Operations Research*, 6 (1): 113–121.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2006). *Actuarial theory for dependent risks: measures, orders and models*. John Wiley & Sons.

Eugene, A., Novita, M., and Nurrohmah, S. (2021). Gini shortfall: A gini mean difference-based risk measure. In *Journal of Physics: Conference Series*, vol. 1725, 012094. IOP Publishing.

Filippi, C., Guastaroba, G., and Speranza, M. (2017). Applications of conditional value-at-risk beyond finance: a literature review. In *Technical Report 2*.

Furman, E., Wang, R., and Zitikis, R. (2017). Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks. In *Journal of Banking & Finance*, 83: 70–84.

Gifford, S. (2003). Risk and uncertainty. In *Handbook of entrepreneurship research*, 37–53. Springer.

Giorgi, G.M. and Crescienzi, M. (2001). A look at the bonferroni inequality measure in a reliability framework. In *Statistica*, 61 (4): 571–583.

Kisiala, J. (2015). Conditional value-at-risk: Theory and applications. In *arXiv preprint arXiv:1511.00140*.

Klefsjö, B. (1984). Reliability interpretations of some concepts from economics. In *Naval research logistics quarterly*, 31 (2): 301–308.

Lorenz, M.O. (1905). Methods of measuring the concentration of wealth. In *Publications of the American statistical association*, 9 (70): 209–219.

Panjer, H.H., Boyle, P.P., Cox, S.H., Dufresne, D., Gerber, H., Mueller, H., Pedersen, H., Pliska, S., Sherris, M., Shiu, E., et al. (1998). *Financial Economics: With Applications to Investments, Insurance, and Pensions*. Actuarial Foundation Schaumburg, Ill.

Pflug, G.C. (2000). Some remarks on the value-at-risk and the conditional value-at-risk. In *Probabilistic constrained optimization*, 272–281. Springer.

Righi, M.B. (2019). A composition between risk and deviation measures. In *Annals of Operations Research*, 282 (1): 299–313.

Rockafellar, R.T. and Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. In *Journal of banking & finance*, 26 (7): 1443–1471.

Rockafellar, R.T., Uryasev, S., et al. (2000). Optimization of conditional value-at-risk. In *Journal of risk*, 2: 21–42.

Sarykalin, S., Serraino, G., and Uryasev, S. (2008). Value-at-risk vs. conditional value-at-risk in risk management and optimization. In *State-of-the-art decision-making tools in the information-intensive age*, 270–294. Informs.

Singpurwalla, N.D. (2007). Reliability and survival in financial risk. In *Advances In Statistical Modeling And Inference: Essays in Honor of Kjell A Doksum*, 93–114. World Scientific.

Wang, S.S. (2004). Cat bond pricing using probability transforms. In *Geneva Papers: Etudes et Dossiers*, 278: 19–29.