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GALOIS LATTICE AND POSITIONAL DOMINANCE

Maria Mircea[1], **Jürgen Pfeffer**
Bavarian School of Public Policy, Technical University of Munich, Munich, Germany

Abstract *Galois lattices can be applied to analybinary two-mode networks, contain-ing actors and affiliations. Of particular interest is how they help visualize hierarchical structures in the data by using algebraic set theory. This paper aims to outline the concept and definitions as well as to give insights into the inner working of Galois lattices. Furthermore, we discuss how Galois lattices are related to the recently introduced concept of positional dominance, which describes a relation between two nodes based on their neighborhoods. We demonstrate that by utilizing a reduced labeling approach, all paths from the global lower bound to the global upper bound of a Galois lattice determine exactly the definition of positional dominance applied on two-mode networks. By comparing path lengths starting at either of the bounds, hierarchical levels can be identified. Hence, we conclude that a Galois lattice describes positional dominance and hierarchical levels among actors and respectively among affiliations. We propose two algorithms, one to build a Galois lattice and another to extract the positional dominance from a reduced labeled Galois lattice.*

Keywords: *Galois lattice, Positional dominance, Two-mode networks.*

1. INTRODUCTION

In order to draw conclusions from the analysis of complex networks, it is essential to determine groups and to identify hierarchical levels. The concept of lattices and especially those with a Galois connection have drawn a lot of attention. Many definitions and theorems were formulated in this regard resulting in a theoretical framework for lattice analysis called Formal Concept Analysis. One of the first books on Formal Concept Analysis is Ganter and Wille (1999), which describes algebraic structures and conclusions. The power of Galois lattices is that they order a network into a convenient structure to reveal groups and hierarchies. In general, Galois lattices follow algebraic set theory, which is defined through the neighborhood of nodes in a two-mode network. Applications and the general concept of building Galois lattices from binary data can be found in Ganter et al. (2005). Moreover, these formal concepts have been found to be applicable

[1] Corresponding author: Maria Mircea, email: maria.mircea@tum.de

in the context of network analysis. The first application to network analysis was by Freeman and White (1993), who explained how Galois lattices can be used to analyze binary two-mode networks. In addition, the theories were expanded for application to networks by, for example, clustering the Galois lattice to obtain further insight into the network structure. We adhere to the notations and definitions laid out in the preceding paper throughout. Further extensions for Galois lattices were introduced; the most important ones include the paper of Freeman (1996), where cliques were used before applying the concept of Galois lattices in order to facilitate the visualization. In Kuznetsov (2007) the definition of stability in a Formal Concept was discussed and in Lehmann and Wille (1995) the Galois lattice theory is extended from two-mode networks to three-mode networks.

In this paper, we present the basic theory of Galois lattices and their relationship to binary two-mode networks. In order to formalize the hierarchy displayed by Galois lattices the definition of positional dominance is stated and the connection to Galois lattices is shown. Furthermore, we propose two algorithms, one to build a Galois lattice and another to extract positional dominance from a directed reduced labeled Galois lattice. Both algorithms will be applied to networks to show the inner working of the theory.

2. GALOIS LATTICE

In order to introduce the concept of Galois lattices, some definitions are necessary. The notations and expressions in Freeman and White (1993) will be used. In the theory of formal concept analysis the nomenclature is different (see Ganter and Wille, 1999) but the substance remains the same. The following three definitions are needed to understand the concept of Galois lattices.

Definition 1 (Partially Ordered Set) *A partially ordered set is a set $X \neq \emptyset$ with relation $\leqslant$ such that*

- $\alpha \leqslant \alpha \;\; \forall\, \alpha \in X$ *(reflexivity)*
- $\alpha \leqslant \beta$ *and* $\beta \leqslant \alpha$ *then* $\alpha = \beta \;\; \forall\, \alpha, \beta \in X$ *(antisymmetry)*
- $\alpha \leqslant \beta$ *and* $\beta \leqslant \gamma$ *then* $\alpha \leqslant \gamma \;\; \forall\, \alpha, \beta, \gamma \in X$ *(transitivity)*

In the remainder the relationship $\leqslant$ is given by subsets, i.e. $X \leqslant Y$ if $X \subset Y$.

Definition 2 (Meet and Join) *Given a pair of elements $x, y \in X$ an element m is a lower bound if $m \leqslant x$ and $m \leqslant y$. A lower bound m is called the greatest lower*

bound (or meet) if there is no element b such that $b \leqslant x$, $b \leqslant y$ and $m \leqslant b$.
An element n is called upper bound if, for $x, y \in X$ it holds that $x \leqslant n$ and $y \leqslant n$. The least upper bound (or join) is an element n such that there is no element b with $x \leqslant b$, $y \leqslant b$ and $b \leqslant n$.

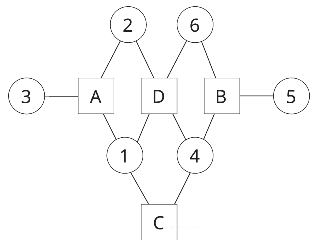
Definition 3 (Lattice) *A lattice is a partially ordered set $(X, \leqslant)$ in which every pair $x, y \in X$ has a meet and a join.*

These concepts will be directly defined and applied to the theory of networks, which is given as follows in Brandes (2016).

Definition 4 (Network) *A network is a mapping $x : S \to W$ assigning values in a range W to dyads from a finite domain $S \subset N \times A$ comprised of ordered pairs of nodes N and affiliations A.*
If $N \cap A = \emptyset$, then S is called an affiliation domain and x a two-mode network.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a) The binary two-mode data matrix of the network represented in Figure 1b



(b) Two-mode graph representation. The edges represent the values W with 1 if there is an edge between two vertices.

Fig. 1: Two-mode network with nodes *N:* {1, ..., 6} and affiliations A: $\{A, \ldots, D\}$.

Another notation for a network is $x \in W^S$, where S is a set of two dimensional indices. W takes values in $\{0, 1\}$ indicating that a node $n \in N$ either has a connection to an affiliation $a \in A$ or not. The concept of a Galois lattice holds for binary two-mode networks, thus the set of nodes N and affiliations A are disjoint sets, i.e. $N \cap A = \emptyset$ and $A \cup N = S$, spanning the entire domain of the network image. Note that Galois lattices can also be applied to one-mode networks, resulting in a symmetric Galois lattice. In Figure 1b an example for a binary two-mode network can be seen. This example serves as a reference for the forthcoming applications and theories. The nodes and hence the set N of the network are in this example $1, \ldots, 6$

and the affiliations A are $A,...,D$. If a relation exists, such that $(a,n) \in \{1\} \subset W$, then this is indicated by an edge.

Consider the two following mappings that are defined on the power set of N, i.e. the set of all subsets of N and the power set of A, denoted by $\mathscr{P}(N)$ and $\mathscr{P}(A)$.

$$\begin{aligned} \phi_1 : \mathscr{P}(\mathscr{A}) &\to \mathscr{P}(N) \\ a &\longmapsto \phi_1(a) := \{n \in N : (n,a) \in W \ \ \forall a \in \mathscr{P}(A)\}, \end{aligned} \tag{1}$$

and

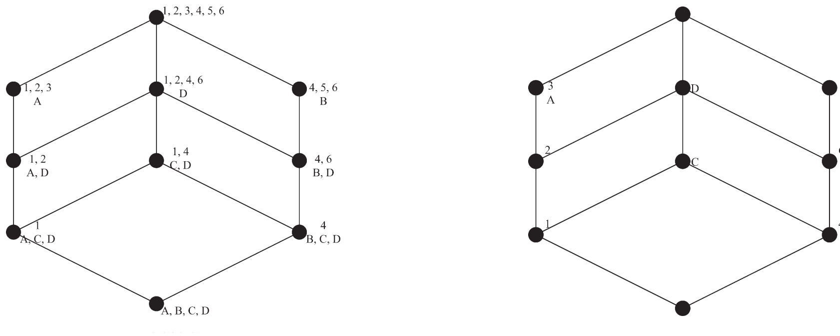
$$\begin{aligned} \phi_2 : \mathscr{P}(N) &\to \mathscr{P}(A) \\ n &\longmapsto \phi_2(n) := \{a \in A : (n,a) \in W \ \ \forall n \in \mathscr{P}(N)\} \end{aligned} \tag{2}$$

where the relation $(n,a) \in W$ indicates that $(n,a) \in \{1\}$ for a binary network. The images of the mappings consist of different sets of nodes and affiliations defined by the binary relation of the network.

Denote by $|.|$ the number of elements in a set. By construction it holds that $\forall\ i \in \{1,...,|\phi_1| = |\phi_2|\}$ there exist an image of the first mapping $\phi_1^i := \phi_1(n)$ for some $n \in \mathscr{P}(N)$ such that the image of the inverse mapping $\phi_1^{i^{-1}}$ is equal to the image of the second mapping $\phi_2^i := \phi_2(a)$ for some $a \in \mathscr{P}(A)$, such that $\phi_1^{i^{-1}} = \phi_2^i$, because the mappings are defined on the same relation W. Therefore, the index of the images of the mappings are in the remainder superscripted by the same i for ϕ_1^i and ϕ_2^i meaning that the image of ϕ_1^i is equal to the inverse image of ϕ_2^i and vice versa.

Definition 5 (Galois Lattice) *A Galois lattice is a lattice $(X, \leqslant)$ with $X \subset \mathscr{P}(N) \cup \mathscr{P}(A)$ whose elements are defined through the images of the two mappings (1) and (2) above on a two-mode network. The pairs (ϕ_1^i, ϕ_2^i) are the elements of the Galois lattice.*

The partial order $\leqslant$ in a Galois lattice is defined by subsets. It holds that $\phi_1^i \leqslant \phi_1^j$ if and only if $\phi_1^i \subset \phi_1^j$ and by construction it holds at the same time that $\phi_2^i \geqslant \phi_2^j$ by the same definition $\phi_2^j \subset \phi_2^i$. In Figure 2a the Galois lattice representation of the binary two-mode network in Figure 1b is displayed. The elements of the Galois lattice are the labels that can be seen on the nodes, where the upper label is the image of Mapping (1) containing different labels of the nodes $1,...,6$ and the lower label is the image of the second Mapping (2) with different labels of affiliations $A,...,D$.



(a) The Galois lattice of the network in Figure 1b with the results of ϕ_1 on the first line of the vertex label and the results of the second mapping ϕ_2 in the second line

(b) The same Galois lattice as in Figure 2a with reduced label for the vertices.

Fig. 2: Galois lattice

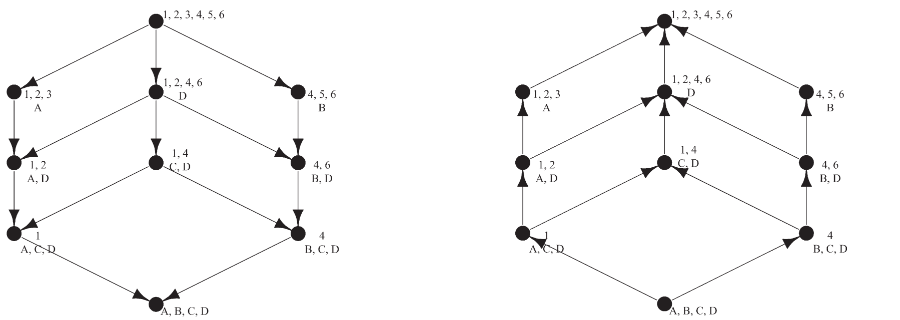
2.1. PATHS IN A GALOIS LATTICE

In order to introduce hierarchies in a Galois lattice we use directed paths and follow the definition introduced in Hennig et al. (2012).

Definition 6 (Path) *Let $s, t \in N$ then a sequence of edges $((n_i, a_j) \in \{1\})$ $(n_0, a_0), (a_1, n_1), (n_1, a_2), ..., (a_k, n_k)$ with $n_0 = s$ and $n_k = t$ is called a directed st-path*

For a Galois lattice it is essential to define directions in order to look for paths, because there exist two directions depending on the mapping in consideration. The structure of a Galois lattice is predefined by the definition of a lattice, which orders its elements according to their meet and join. For simplicity we say that a Galois lattice has the direction of the first mapping, when the arrow indicating the direction goes from two elements of the first mapping ϕ_1^i and ϕ_1^j towards their meet, see Figure 3a. At the same time, the arrow is pointing towards the join defined by Mapping (2). The other direction, in Figure 3b, is exactly opposite to the previous one and we say that the Galois lattice has the direction of the second mapping; the arrows follow now the meet of elements ϕ_2^i and ϕ_2^j and similarly the join of Mapping (1).

A Galois lattice is defined by relation W, hence the directions are exactly opposite to each other and the analysis can be done for either of the mappings by keeping in mind which one was chosen. Therefore, without loss of generality,



(a) The Galois lattice with direction defined by the first mapping ϕ_1

(b) The Galois lattice with direction defined by the second mapping ϕ_2

Fig. 3: Directed Galois lattices

in the remainder we consider the direction defined by the second mapping unless mentioned otherwise. We say the global upper bound is the uppermost node, where the arrows are pointing towards it, respectively, the global lower bound is the bottommost node, where arrows are pointing away from. As in a Galois lattice every element has a meet and a join, a global upper and lower bound always exist. Consequently, the global upper bound contains all elements of N as long as they have a connection to any affiliation, and the global lower bound contains all elements of A as long as they have a connection to any node.

2.2. LABEL OF A GALOIS LATTICE

A *label* is defined as the elements of the two images $\phi_1^i = \{n_j\}_{j \in I_1}$ and $\phi_2^i = \{a_k\}_{k \in I_2}$. A *reduced label*, denoted by (N^i, A^i) consists of the subsets $N^i \subset \phi_1^i$ and $A^i \subset \phi_2^i$. It holds that for $a \in A^i$ it has the names of those affiliations for which it is the least element containing those affiliations, with respect to the second direction. Furthermore $n \in N^i$ has the names of those nodes for which it is the greatest element containing these nodes.

Meaning that for an element (ϕ_1^i, ϕ_2^i) with $\phi_2^i = \{a_k\}_{k \in I_2}$ the label is reduced to those a_k such that for $\phi_1^i = \{n_j\}_{j \in I_1}$, a_k has a relation $(a_k, n_j) \in \{1\}$ with all n_j and only with those n_j. Equally only those n_j remain in the label that have exactly to all $a_k \in \phi_2^i$ and just to these a_k a relation $(a_k, n_j) \in \{1\}$. Thus the label of the Galois lattice becomes very sparse and has elements without any label as can be seen in Figure 2b.

3. POSITIONAL DOMINANCE

The definitions follow the concepts introduced in Brandes (2016), which are motivated by past attempts to define positions in social networks and are a result of finding a general concept of position combining all previous theories. For example the following ideas of positions have been adapted to find a generalized definition for position in social networks. In Borgatti and Everett (1992) two notions of positions are compared to each other. The structural equivalence, which defines the position of nodes based on their neighborhood and the structural isomorphism, which defines a permutation of the nodes, allowing to distinguish positions of the nodes based on their degree. In Breiger and Pattison (1986) positions are defined by algebraic structures. Furthermore, the concept is derived from Blau (1977), where the concepts of positions in social space is discussed and combined with McPherson (1983), where the macro-sociological theory is expanded by micro-structural dynamics. These ideas of positions in social networks can all be summarized with the following definition by Brandes (2016):

Definition 7 (Network Position) *Given a network $x \in W^S$ on a domain $S \subset N \times A$, the position of $i \in N$ in x is defined as*

$$pos(i|x) = \{x : i \to t : (i,t) \in S\}. \tag{3}$$

For example, for the graph in Figure 1b, the position of a node, e.g. node 2, is defined through the connections it has to its affiliations. Thus, the position is defined by the presence of connections with A and D and the absence of connections with the other affiliations.

Definition 8 (Positional Dominance) *Let $x \in W^S$ be a network on a dyadic domain $S \subset N \times A$ with values in a range W that is preordered by $\leqslant$. For $i, j \in N$, we say that $p(i|x)$ is dominated by $p(j|x)$, denoted by $p(i|x) \leqslant p(j|x)$, if there exists a permutation $\pi : A \to A$ such that for every $(i.t) \in S$, we have*

$$(j, \pi(t)) \in S \;\; with \;\; x_{it} \leqslant x_{j\pi(t)}. \tag{4}$$

If the identity is used for the permutation π, which will be the case in the remainder, then for the graph in Figure 1b node 1 positionally dominates node 2, because node 1 has connections $\neq 0$ with at least the affiliations of node 2, namely A, C and D, whereas node 2 just has connections $\neq 0$ with A and D.

There exist algorithms to calculate positional dominance for networks (Brandes et al., 2017). The algorithms presented in the paper for positional dominance are not limited to binary data as restraint here. The complexity of their algorithms is $\mathcal{O}(nmloglog\Delta(G))$, which is an improvement to the strait forward approach.

In Schoch and Brandes (2016), it is shown how centrality measures can be used in a common framework based on path algebras by using positional dominance to show that neighborhood inclusion is preserved by centralities.

For the purpose of Galois lattices, we expand the theory of positional dominance to different levels, i.e. the first level of positional dominance consists of those nodes which are not dominated by any other node. The second level of positional dominance are those nodes that are only dominated by nodes in the first level of positional dominance and so on.

Definition 9 (Levels of Positional Dominance) *Let $x \in W^S$ be a network on a dyadic domain $S \subset N \times A$ with values in a range $\{0,1\}$ that is preordered by $\leqslant$. We say $i \in N$ is in the* ***first level of positional dominance*** $p_1(i|x)$*, if*

$$\nexists\ j \in N \ \ with \ \ p(i|x) \leqslant p(j|x). \tag{5}$$

We say $i \in N$ is in the ***kth level of positional dominance*** $p_k(i|x)$*, if for a $j \in p_{k-1}(j|x)$, $k-1$ is the maximal level of positional dominance such that it holds*

$$j \in N \ \ with \ \ p_k(i|x) \leqslant p_{k-1}(j|x). \tag{6}$$

An illustration of the concept can be seen in Figure 1b in the level sequence of nodes $1,2$ and 3. As already shown, node 1 positionally dominates node 2 and with the same derivations node 2 positionally dominates node 3. Therefore, node 1 is in the first level of positional dominance because there is no other node positionally dominating it. Node 2 is in the second level of positional dominance because it is just dominated by nodes from the first level, i.e. node 1. Notice that node 2 is not dominated by any other node on a higher positional dominance level in the network. Hence, with the same arguments, node 3 is in the third level of positional dominance.

4. CONNECTON BETWEEN GALOIS LATTICES AND POSITIONAL DOMINANCE

Before the definition of positional dominance stated in Brandes (2016), there have been definitions of dominance in networks, in particular for directed graphs. The first definition of dominance in a directed graph was by Prosser (1959), who defines a dominating node by involvements in paths of other nodes. Attempts to find an efficient algorithm for this definition can be seen for example in Cooper et al. (2006) and Georgiadis et al. (2006). However, this type of dominance is not applicable to undirected networks, which is the case here. Thus, in Gamble et al. (2016) a definition can be found with regard to dominance in an undirected graph. The node dominance is defined by the neighborhood of a node, saying that if the neighborhood of a node W is a subset of the neighborhood of a node v, then v dominates w. The definition in Brandes (2016) is a more general definition of the above, but still applicable in the case of Galois lattices. Hence, a Galois lattice represents the node dominance for a binary two-mode network.

We show that directed paths in a Galois lattice correspond to the definition of positional dominance in Brandes (2016), one labeled node n_l positionally dominates another node n_j, if the set of affiliations a that have a relation $(n_l, a) \in \{1\}$ with node n_l is a superset of the set of affiliations that have a relation $(n_j, a) \in \{1\}$ with node n_j, which will be shown next.

Denote by $N^i = \{n_j\}_{j \in I_1}$ and $A^i = \{a_k\}_{k \in I_2}$ the reduced label of an element (ϕ_1^i, ϕ_2^i) of the Galois lattice. Furthermore, the set of all affiliations is denoted by A and the set of all nodes by N.

First, consider two elements with reduced labels (N^i, A^i) and (N^j, A^j) such that $N_i \neq \emptyset$ and $N_j \neq \emptyset$. Without loss of generality assume that there exists a directed path from (N^j, A^j) to (N^i, A^i) in a Galois lattice defined by Mapping (1). The direction defined by the first mapping is pointing towards the meet of the elements of the first mapping and the join of the elements of the second mapping. Denote by $(\phi_1^k, \phi_2^k)_{k \in K}$ the nodes visited in the path. Let the index set $K = \{k^1, \ldots, k^n\}$ be ordered according to the position in the path starting in (N^j, A^j), where n is the total number of visited nodes. Hence, in the first step from (N^j, A^j) to $(\phi_1^{k^1}, \phi_2^{k^1})$ at least one element $a \in A$ is added to the set $\phi_2^{k^1}$, because the direction is pointing towards the join of the second mapping. Recursively this can be done for every step $k^i \in K$ on the path. Consequently, by the definition of the relation $\leqslant$ in a Galois lattice it follows that $\phi_2^j \subset \phi_2^i$. Furthermore, for $n^i \in N_i$ it holds by the definition of a reduced label that it has a relation $(n^i, a_k^i) \in \{1\}$ with all $a_k^i \in \phi_2^i$ and only with the elements of ϕ_2^i and otherwise zero relations. The same holds for $n^j \in N_j$ with the elements of ϕ_2^j. Consequently, the set of affiliations of n^i is a superset of the set of affiliations of n^j, which is the relationship that had to be shown for positional dominance. Thus, with $\phi_2^j \subset \phi_2^i$ it follows from the definition of positional dominance that $p(n^i|x) \leqslant p(n^j|x)\ \forall\, n^i \in N_i$ for the network x and the notation of positional dominance.

With the same arguments using the direction defined by the second mapping, information about the positional dominance of the affiliations is obtained. Therefore, the positional dominance relation between two nodes in a binary two-mode network can be found by looking, for either of the two directions, for all paths from one node to the other. In general, if we look for all paths from the global lower bound to the global upper bound, all positional dominance relations between nodes, and respectively affiliations, can be found.

5. ALGORITHMS

Different algorithms were proposed to construct a Galois lattice. One of the first algorithms is described in Ganter and Kuznetsov (1998) where two basic algorithms are given, one to produce all closed sets of a closure operator and one to compute the minimal families of implications. Finding all concepts of a lattice has often high complexity as stated in Kuznetsov and Obiedkov (2002), where different algorithms are compared according to their theoretical and experimental performance. Their results state that for finding all concepts the algorithm suggested in Lindig (2000) gives good results, while Norris (1978) is recommended for the whole lattice. In Nourine and Raynaud (1999) an algorithm is suggested that computes the covering graph of a lattice, improving over Bordat (1986) and Ganter (2010), where a stepwise construction of the Galois lattice is suggested.

5.1. CONSTRUCTING A GALOIS LATTICE

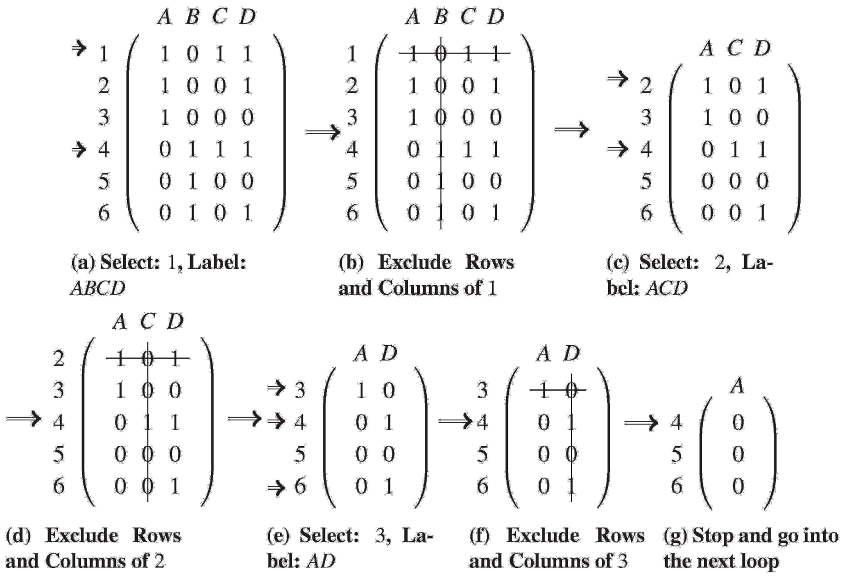
We propose a recursive algorithm to find the Galois lattice graph. It operates on the incidence matrix of the graph and once a node is found the matrix is reduced to find the neighboring nodes in the Galois lattice. The procedure can be seen in Algorithm 1, where the input is an incidence matrix X and the output a Galois lattice G. The algorithm can be used to create both the different subsets and the covering graph, for either of the two mappings. It uses the rows to define the algebraic subsets and thus creates from the names of the columns the Galois lattice. The procedure results in a Galois lattice with the labeling of only one mapping. The other corresponding mapping can be obtained by finding for each element of the current mapping its affiliations and intersecting them. The algorithm starts in the global lower bound, defined by the rows and recursively finds neighboring nodes, thus the algorithm can additionally be applied to produce a directed Galois lattice, e.g. in order to find the positional dominance path.

Figure 4 shows one loop of the algorithm. In Figure 4a) the rows 1 and 4 are selected due to their maximal row sum. Then, the algorithm checks if the other rows are covered by those two rows, which is true. Hence, no other row is selected. In the first loop row 1 is selected by the sequential order. In Figure

Algorithm 1 Galois Lattice

Input: X a incidence matrix
Output: G a Galois Lattice

```
G ← Graph: n_1 := colnames(X)
function FIND_GALOIS_NODES(X)
    selected ← [ ]
    sub ← [ ]
    while selected ∪ sub ≠ colnames(X) do
        selected ← selected ∪ max(rowSums(X))
        for i in selected do
            sub ← sub ∪ ∀ rows which have 0 at the same columns as selected[i]
        end for
    end while
    if dim(X) > 0 and ∃ non-zero entries then
        for i ∈ selected do
            frontier ← colnames(X)
            labelX ← colnames(X[,where selected[i] ≠ 0])
            if (labelX ∈ G) ∨ (labelX ∈ frontier) then
                add edge to G: labelX ↔ colnames(X)
            else
                add vertex to G: labelX
                add edge to G: labelX ↔ colnames(X)
            end if
            newX ← X[−selected[i], −colnames(X) where selected[i] has zero entries]
            find_galois_nodes(newX)
        end for
    end if
    return G
end function
```



(a) Select: 1, Label: *ABCD*
(b) Exclude Rows and Columns of 1
(c) Select: 2, Label: *ACD*
(d) Exclude Rows and Columns of 2
(e) Select: 3, Label: *AD*
(f) Exclude Rows and Columns of 3
(g) Stop and go into the next loop

Fig. 4: Procedure of Algorithm 1

4b) the selected row and the columns with zero entries for row *B* are excluded for the recursive algorithm. Figure 4c) shows the next rows selected for the smaller matrix. The loop terminates in Figure 4g) with an empty matrix.

5.2. EXTRACTING POSITIONAL DOMINANCE FROM A GALOIS LATTICE

The second algorithm we propose uses the results discussed in Section 3. Consequently, it searches for the paths from one node to another in a directed reduced labeled Galois lattice in order to identify the positional dominance relation between these nodes. First, the algorithm searches for all possible directed paths between the specified nodes. Then the labels of the considered nodes, e.g. all nodes or all affiliations, are intersected with the nodes found on the path, such that nodes without label and nodes without relevance are excluded. In the end a graph is created from all paths found in this procedure, see Algorithm 2.

5.3. GALOIS LATTICE LAYOUT

An important step is to find an appropriate visualization to easily identify groups and hierarchical structures. Hence, we define a layout using the relationship between positional dominance and Galois lattices. In order to identify hierarchical positions in a Galois lattice we look at the paths of both mappings defined as $d1$ and $d2$, which include all distances from the global lower bound to all other nodes

Algorithm 2 Positional Dominance

Input: G a Galois Lattice, $Names$ a vector of names
Output: T a Graph describing positional dominance

```
function FIND_POSITIONAL_DOMINANCE(X)
    D ← find_all_paths(G, from = lowerBound, to = upperBound)
    T ← empty_graph, n_1(T) ← name_of(lowerBound)
    for path ∈ D do
        path ← Names ∩ path
        for node ∈ path do
            T ← add_vertex(node_i)
            T ← add_edge(node_{i−1}, node_i)
        end for
    end for
    return T
end function
```

for $d1$ and all distances from the global upper bound to all other nodes in $d2$. If a Galois lattice is created for the first mapping then the positions of the nodes on the $y-$axis are defined as follows. Assume that the nodes in a Galois lattice are the points x_i, then

$$pos(x_i) = \frac{d1_i}{d1_i + d2_i} \cdot max(d1), \quad \forall i \in [n].$$

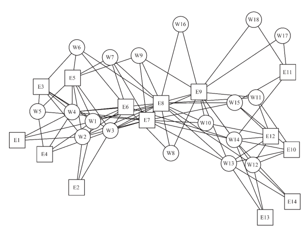
Consequently, the different levels of positional dominance can be seen. The nodes, considering the direction of the first mapping, are placed highest, directly at or after the global upper bound. The second level of positional dominance can be identified by following the y-axis to the next labeled nodes from the first level of positional dominance nodes.

For the $x-$axis a force directed algorithm is used that arranges nodes with respect to their common edges. Here the algorithm of *Fruchterman-Reingold* was used, which can be found in Fruchterman and Reingold (1991). As a consequence of the force directed algorithm groups can be identified along the $x-$axis by identifying their membership. In a Galois lattice this membership is defined by the meets and joins of each two nodes. For this reason nodes with more common affiliations are placed closer to each other.

With the help of this layout it is sufficient to differentiate group structures in the nodes along the x-axis in the first group of positional dominance defined

		E1	E2	E3	E4	E5	E6	E7	E8	E9	E10	E11	E12	E13	E14
W1	Mrs. Evelyn Jefferson	1	1	1	1	1	1	0	1	1	0	0	0	0	0
W2	Miss Laura Mandeville	1	1	1	0	1	1	1	1	0	0	0	0	0	0
W3	Miss Theresa Anderson	0	1	1	1	1	1	1	1	1	0	0	0	0	0
W4	Miss Brenda Rogers	1	0	1	1	1	1	1	1	0	0	0	0	0	0
W5	Miss Charlotte McDowd	0	0	1	1	1	0	1	0	0	0	0	0	0	0
W6	Miss Frances Anderson	0	0	1	0	1	1	0	1	0	0	0	0	0	0
W7	Miss Eleanor Nye	0	0	0	0	1	1	1	1	0	0	0	0	0	0
W8	Miss Pearl Oglethorpe	0	0	0	0	0	1	0	1	1	0	0	0	0	0
W9	Miss Ruth DeSand	0	0	0	0	1	0	1	1	1	0	0	0	0	0
W10	Miss Verne Sanderson	0	0	0	0	0	0	1	1	1	0	0	1	0	0
W11	Miss Myra Liddell	0	0	0	0	0	0	0	1	1	1	0	1	0	0
W12	Miss Katherine Rogers	0	0	0	0	0	0	0	1	1	1	0	1	1	1
W13	Mrs. Sylvia Avondale	0	0	0	0	0	0	1	1	1	1	0	1	1	1
W14	Mrs. Nora Fayette	0	0	0	0	0	1	1	0	1	1	1	1	1	1
W15	Mrs. Helen Lloyd	0	0	0	0	0	0	1	1	0	1	1	1	0	0
W16	Mrs. Dorothy Murchison	0	0	0	0	0	0	0	1	1	0	0	0	0	0
W17	Mrs. Olivia Carleton	0	0	0	0	0	0	0	0	1	0	1	0	0	0
W18	Mrs. Flora Price	0	0	0	0	0	0	0	0	1	0	1	0	0	0

(a) Incidence matrix



(b) Two-mode network representation

Fig. 5: Southern women data set

by the *y*-axis and distribute the nodes on lower levels of positional dominance in accordance to the grouped nodes that positionally dominates them.

6. ILLUSTRATIVE EXAMPLE

In this section we demonstrate the concept of Galois lattices on the southern

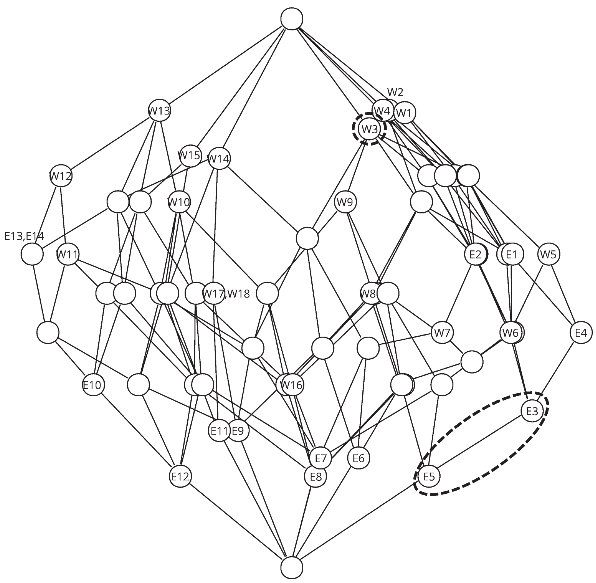


Fig. 6: The Galois lattice of the DGG data

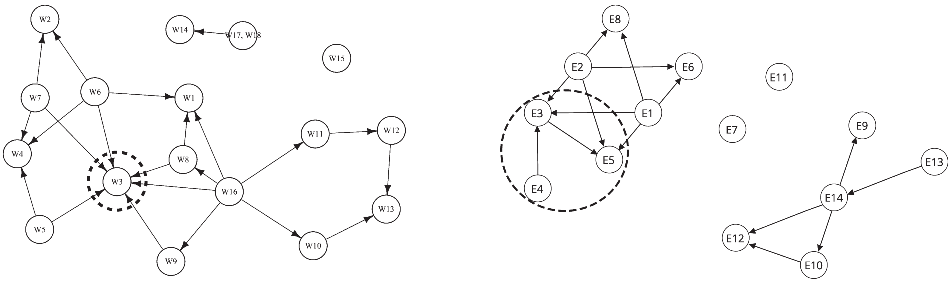
women dataset (Davis et al., 1941), which consists of 18 women attending 14 events. The data was collected for ethnographic studies in Mississippi. The matrix of the data set can be seen in Figure 5a and the corresponding two-mode network is displayed in Figure 5b.

If Algorithm 1 is applied to the data set, then the resulting image can be seen in Figure 6. In the first level of positional dominance, the women are divided into two groups, $\{W1,W2,W3,W4\}$ and $\{W13,W14,W15\}$, which can be seen along the x-axis. Furthermore, the women in the first group of positional dominance span the set of attended events. The other women attended only those events that were attended by the women in the first level. At the second level of positional dominance are also two groups. The first group of women is $\{W5,W6,W7,W8,W9\}$, which correspond to the first group of women in the first level and the second group $\{W10,W11,W12,W17,W18\}$, which relates to the second group on the first level. These are simply found by following the path down along the y-axis until the next labeled node is found, such that these women are positionally dominated by the women in the first group and only by those. Note that $W17$ and $W18$ have attended the exact same events, hence they share a node in the Galois lattice. $W16$ is on the last level of positional dominance and is dominated by women from both groups.

Respectively, this analysis can be done for the events, taking the reverse direction into consideration. Events in the highest level of positional dominance are at the bottom of the Galois lattice and can be separated into three groups $\{E9,E11,E12\}$, $\{E6,E7,E8\}$ and $\{E5\}$. The set relations are especially clear for the group starting in $\{E5\}$, because a path of direct inclusions exists. Since $\{E3\}$ is directly connected to $\{E5\}$ this implies that all women who attended $\{E3\}$ also attended $\{E5\}$. Moreover, $\{E3\}$ is directly connected to $\{E4\}$, which results in an inclusion chain, i.e.

$$\{E4\} \hat{=} \{W1,W3,W4,W5\} \subset \{E3\} \hat{=} \{W1,...,W6\} \subset \{E5\} \hat{=} \{W1,...,W7,W9\}.$$

In order to show that the above conclusions about positional dominance hold true, Algorithm 2 is applied to directly see the positional dominance relation. Hence, applying the Algorithm 2 to the reduced labeled directed Galois lattice results in Figure 7. In Figure 7a, the direct positional dominance relation is displayed for the women. The arrows are pointing towards the dominating node. Every group of connected nodes is in no positional dominance relation with other separately connected groups. For the women there are three groups of positional dominance relations. For example, $\{W3\}$ dominates directly six women and has a central position. This can be seen also in the Galois lattice as $\{W3\}$ is in the first level of positional dominance, i.e. at the upper most position in the Galois lattice.



(a) Women (b) Events

Fig. 7: Positional dominance of DGG data

Moreover, $\{W16\}$ is dominated by two groups of women, who do not share other positional dominance relations. Note that these two groups, which are connected by $\{W16\}$, also appear in the analysis of the Galois lattice; but additionally the women $\{W15, W14, W17, W18\}$ were assigned to the group on the right of $\{W16\}$ in the Galois lattice. Even though these women do not share a positional dominance in relation to that group, they still have many events in common. Thus, Galois lattices show additionally different features in the data set, because every two nodes have a meet and a join. Thus, nodes tend to be closer in the layout which have more affiliations in common, even though the set of affiliations are not subsets, as required for positional dominance.

For the events in Figure 7b four groups of positional dominance exist. In particular the relationship between the events $\{E5, E3\}$ and $\{E4\}$ can be seen also here, likewise to the example discussed in the Galois lattice. Note that $\{E4\}$ is not only positionally dominated by $\{E3\}$, but also by $\{E5\}$. As $\{E4\}$ is a direct subset of $\{E3\}$, the dominance of $\{E5\}$ is always induced by $\{E3\}$.

6.1. PREVIOUS METHODS USED ON THE SOUTHERN WOMEN DATA SET

An analysis of this data has been done in a variety of studies, many of which have been summarized in the meta analysis of Freeman (2003). Most of the papers find the same two main groups described here, but differ in their inner order. In Bonacich (1978) also algebraic structures were used. He defines an algebraic difference between three events and groups the women accordingly to their attendance at the events with the smallest symmetric difference. He finds events $E3, E8$ and $E12$ to be optimal for spanning the homomorphism which results in six distinct groups of women. These groups are exclusively based on the previous mentioned events and do not take into account the relationship to other events.

In Doreian (1979) p-simplexes are defined to analyze the number of commonly shared events among women. A simplex consists of either all events a woman attended or those events that were commonly attended by some number of women. For example, $W1$ and $W3$ are grouped together as they share seven events in common and $W2$ and $W4$ are grouped separately because of their high common share of events. In a Galois lattice the four women are placed in the first level of positional dominance indicating that none of them has attended a subset of the other. Thus, the Galois lattice reveals additional information through its hierarchical structure.

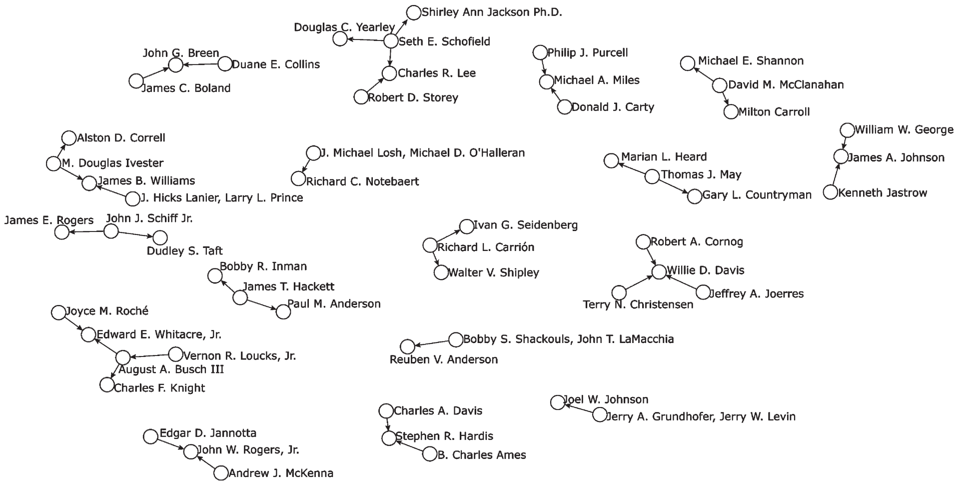
Roberts (2000) normalizes the incidence matrix of the southern women graph and uses correspondence analysis to identify groups. In correspondence analysis the data points are projected onto a lower dimensional manifold and the axes are according to the variance in the data. The data is displayed in two dimensions after the transformation with correspondence analysis. This results in a distinct inner order for the two groups.

Borgatti and Everett (1997) also used a hierarchical approach. First, they defined the notion of an n-biclique for bipartite graphs, i.e. a maximal complete bipartite graph of size n. Based on the overlap of the resulting cliques Johnson's hierarchical cluster was used to determine groups. The core women in the first group result to be $W3$ and $W4$, which are both also in the first level of positional dominance. However, $W1$ and $W2$ are additionally assigned to this group in the Galois lattice, because the groups are formed not only considering their hierarchical structure but also accounting for the number of common events.

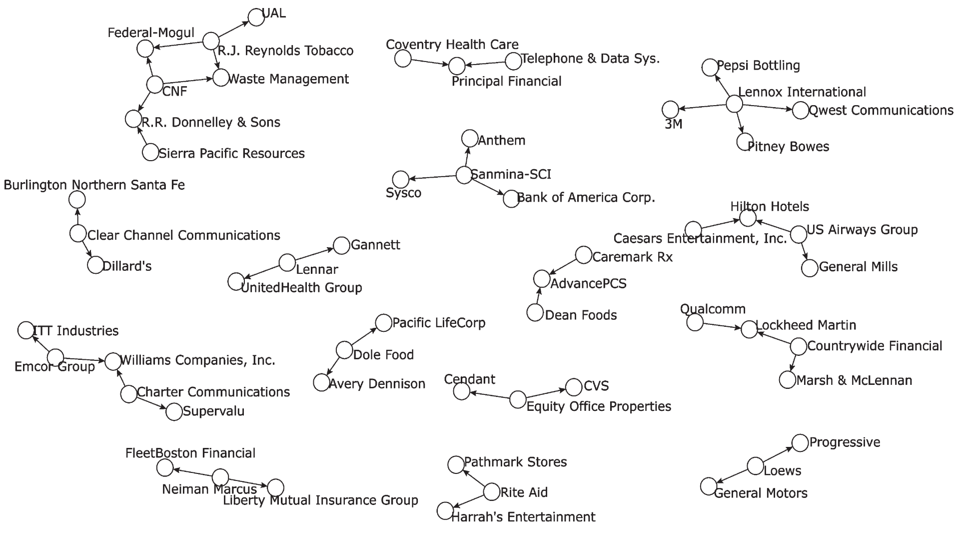
It is clear that the main aspect distinguishing the Galois lattice approach from the other methods is the hierarchal structure amongst the groups and simultaneously ordering the women through the number of common events through the lattice structure.

7. APPLICATION SCENARIO: INTERLOCKING DIRECTORATE

Interlocking directorate networks Levin and Roy (1979) are two-mode networks that consist of managers and companies. An edge connects a manager who serves on a board of a specific company. To demonstrate the algorithms described above, we use data from the US-Fortune 500 companies in 2004 (data source: theyrule.org). This network includes 500 companies, 4,300 managers, and 5,500 edges. We are primarily interested in positional dominance among companies. When extracting the positional dominance networks from the Galois lattice of this network, none of the companies has any dominance connections, because almost 80% of managers in the network are connected to only one company. These network pendants



(a) Managers



(b) Companies

Fig. 8: Positional Dominance Network of Interlocking Directorates

prohibit that a company's managers are a complete subset of another company.

We suggest a relaxed approach for calculating Galois lattice and positional dominance by removing all pendants (degree = 1) from the network. The positional dominance relations from the resulting network with 873 managers and 2,115 edges can be seen in Figures 8a and 8b. It is important to remember that dominance is visualized with an arrow pointing towards the dominator. Fig. 8a illustrates that structural equivalent nodes are merged to one node. For instance, Grundhofer and Levin (lower right corner of the visualization) have edges to the exact same companies and both nodes are dominated by Johnson. Note, that the merging of structurally equivalent nodes is a *side effect* of calculating the Galois lattice.

8. DISCUSSION

We showed that Galois lattices can be used to order a binary two-mode network to illustrate positional dominance. This tool can visualize hierarchical levels in a binary network and identify groups. The advantage of using Galois lattices is that different levels of positional dominance can be easily understood. Especially the node's hierarchical position is immediately noticeable by its position on the y-axis.

The notion of positional dominance is a general definition that holds for many networks; in contrast the tool of Galois lattices presented here is limited to binary data. In Brandes et al. (2017) different efficient algorithms are described to compute positional dominance. These algorithms are constructed for a wider range of networks. However, the advantage of reducing the application of the algorithms to binary data is that the different groups, which share a positional dominance relationship, can be easily identified. These groups do not share common edges. On the contrary, in a Galois lattice every node has an edge connecting it to the whole graph. Furthermore, groups other than positional dominance groups can be found in a Galois lattice, as a lattice is defined through set theory, i.e. every meet and join of the elements in the Galois lattice is present. For this reason nodes with more common meets and joins are placed closer to each other by the force directed layout for the x-axis.

In conclusion, for identifying either the positional dominance between two nodes or the positional dominance between groups that exist in any network, the general algorithms for positional dominance proposed in Brandes et al. (2017) are sufficient. For finding finer levels of positional dominance and groups based on common affiliations with hierarchical levels for binary data, Galois lattices can be more useful.

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