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MIXTURE OF TWO GENERALIZED INVERTED EXPONENTIAL DISTRIBUTIONS WITH CENSORED SAMPLE: PROPERTIES AND ESTIMATION

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Abstract. *Mixture models have significantly been used in survival analysis. This study considers Bayesian analysis of the two-component mixture distribution of generalized inverted exponential distributions. Also its reliability characteristics and important distributional properties are presented. We have considered this particular distribution because it is skewed and is considered appropriate in engineering processes, when an engineer suspects a high failure rate in the beginning, but after continuous inspection, the failures go down. The Bayesian estimation of unknown parameters of the mixture of generalized inverted exponentialdistributions under type-I censoring, assuming two priors is investigated using different loss functions.It is seen that the closed-form expressions for the Bayes estimators cannot be obtained for scale parameter. The efficiencies of the proposed set of estimates of the mixture model parameters are studied through simulation. Posterior risks are evaluated and compared to explore the effect of prior beliefs and loss functions.Simulated results and an example based on a real-life data are also given to interpret the study.*

Keywords: *Reliability function, Hazard rate function, Loss functions, Bayes estimators, Posterior risk.*

1. INTRODUCTION

Mixture models express complex situations than the simpler ones and have been used in almost every fields of statistical sciences to model diverse populations. Mixtures models have been well practiced in many fields such as engineering, economics, marketing, astronomy, psychiatry, medicine, biology etc. Mixture distributions apply when a statistical population contains two or more sub populations. So, mixture densities can be used to model a statistical population with subpopulations. Mixture components are the densities of the subpopulations and weights present the proportion of each subpopulation in the complete population. Sindhu et al. (2014) studied Bayesian analysis of the shape parameter of the mixture

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of Burr type X distribution using the censored data.Gosh and Ebrahimi (2001) have studied the Bayesian analysis of the mixing function in a mixture of two exponential distributions. Saleem and Aslam (2009) presented a comparison study of the maximum likelihood estimates with the Bayes estimates assuming the uniform and the Jeffreys priors for the parameters of the Rayleigh mixture. Sindhu et al. (2014) considered the Bayesian inference for a mixture of Burr type II distribution under type-I censoring. Saleem et al. (2010) considered the Bayesian analysis of the mixture of Power function distribution using the complete and the censored sample. Sultan et al. (2007) investigated the properties of the two component mixture of inverse Weibull distribution under classical approach. With highly realiable components, it is unusual if all the components have failed by the end of the time alloted for the test. When all subjects are scheduled to begin the study at the same time and end the study at the same time type I censoring occurs. Type I censoring is usually used in survival studies and in some engineering studies.

The exponential distribution is the most widely used lifetime model in reliability theory, because of its simplicity and mathematical feasibility.Gupta et al. (1999) considered a three-parameter distribution when the location parameter is not present. Alshingiti (2009) have proposed two parameters generalization invertedexponential distribution.Singh et al. (2013) have proposed the use of IED in survival analysis.Abouammoh and Alshingiti (2009) have discussed many properties and reliability characteristics of generalization invertedexponential distribution. Assuming it is a good lifetime model, they also discussed the maximum likelihood and least square methods for the estimation of the unknown parameters of a generalized inverted exponential distribution. Krishna and Kumar (2012) have studied the reliability estimation based on progressive type-II censored sample under classical setup. They proposed maximum likelihood estimation and least square estimation procedures. Dey and Pradhan (2014) derived maximum likelihood estimators of the unknown parameters and the expected Fisher's information matrix of the generalized inverted exponential distribution and obtained Bayes estimation under the squared error loss function. These Bayes estimates were evaluated by applying Lindley's approximation method, the importance sampling procedure and Metropolis–Hastings algorithm. Oguntunde and Adejumo (2015) proposed a two parameter Inverted Generalized Exponential (IGE) and a three parameter Generalized Inverted Generalized Exponential (GIGE) probability models as generalizations of the one-parameter. They explored the statistical properties of the GIGE distribution and its parameters were estimated for both censored and uncensored cases using the method of maximum likelihood estimation (MLE). Dube et al. (2016) derived maximum likelihood estimators of unknown parameters and reliability characteristics of generalized inverted exponential distribution using progressive first-failure censored samples.

The aim of present study is to investigate the prominent features of the mixture of generalized inverted exponential distributions. This mixture has not been considered earlier in the literature through Bayesian structure to the best of our knowledge. The rest of the paper is organized as follows. In Section 2, we define the mixture model, its properties and likelihood function of mixture of generalized inverted exponential distributions. Inferential procedures with Bayesian estimation are considered for the set of parameters in Section 3, which include the posterior distribution, Bayes estimators and posterior risks under different loss functions. In Section 4, simulation study and comparison of the estimates are given.A real life mixture of generalized inverted exponential distributions is considered in Section 5. Conclusions are reported in Section 6.

2. MIXTURE MODEL AND ITS PROPERTIES

A finite mixture distribution with *m*-component densities of specified parametric form and unknown mixing weight p_1 is given by:

$$f_z(z) = \sum_{i=1}^{m} p_i f_i(z),\ 0 < p_i \le 1,\ \sum_{i=1}^{m} p_i = 1.$$

The presently generalized inverted exponential distribution is supposed for *m*-components of the mixture:

$$f_i(x;\lambda_i,\alpha_i) = \frac{\alpha_i\lambda_i}{x^2} exp\left(-\frac{\lambda_i}{x}\right)\left\{1 - exp\left(-\frac{\lambda_i}{x}\right)\right\}^{\alpha_i - 1};\alpha_i,\lambda_i > 0,\ x > 0 \text{ and } i = 1,2,...,m, \quad (1)$$

where λ and α are scale and shape parameters, respectively. Thus, the said mixture model is of the form:

$$f(x;\Theta) = \sum_{i=1}^{m} p_i\left(\frac{\alpha_i\lambda_i}{x^2} exp\left(-\frac{\lambda_i}{x}\right)\left\{1 - exp\left(-\frac{\lambda_i}{x}\right)\right\}^{\alpha_i - 1}\right);\ i = 1,2,...,m,\ \sum_{i=1}^{m} p_i = 1.$$

The distribution function of the correspondingmixture distribution is:

$$F(x;\ \Theta) = p_1F_1(x;\ \Theta_1) + p_2F_2(x;\ \Theta_2),...,p_mF_m(x;\ \Theta_m)$$

$$= \sum_{i=1}^{m} p_i\left(1 - \left\{1 - exp\left(-\frac{\lambda_i}{x}\right)\right\}^{\alpha_i}\right),\ i = 1,2,...,m,\ \sum_{i=1}^{m} p_i = 1,$$

where $\Theta = \left\{(p_1,p_2,...,p_m),(\lambda_1,\lambda_2,...,\lambda_m),(\alpha_1,\alpha_2,...,\alpha_m)\right\}$, and $\Theta_i = (\lambda_i,\alpha_i), i = 1,2,...,m.$

Graphical representations of different selected parametric values for the mixture model are shown in Fig. 1.

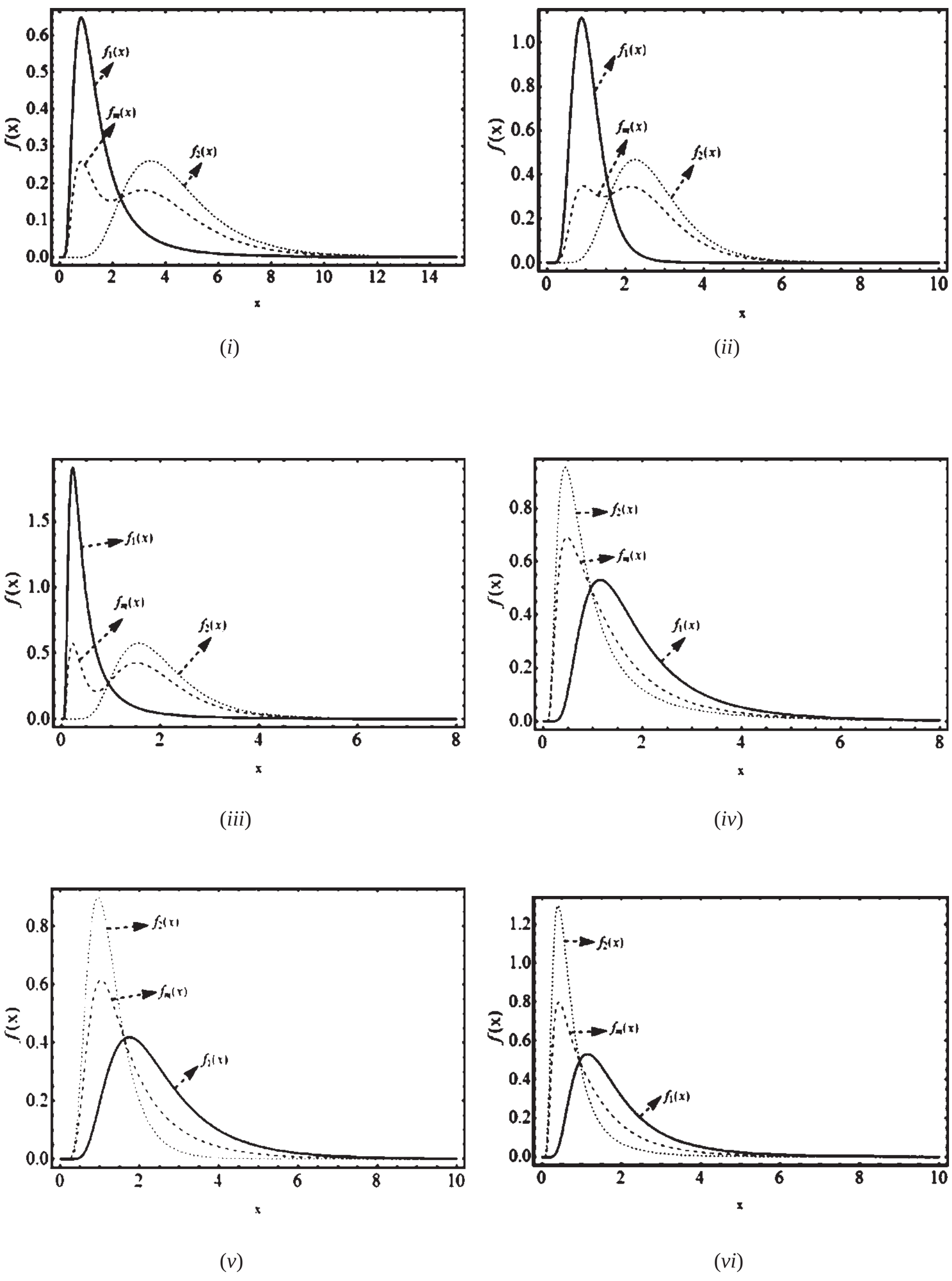


Fig. 1: Density function components and their mixtures $(p_1, \lambda_1, \lambda_2, \alpha_1, \alpha_2)$**(*i*) (0.4, 2, 11, 3, 10), (*ii*) (0.3, 3, 8, 13, 16), (*iii*) (0.3, 0.5, 5, 2, 10), (*iv*) (0.3, 3, 1, 4, 2), (*v*) (0.4, 5, 3, 6, 9) and (*vi*) (0.4, 4, 2,1, 2).**

(a) Reliability function

The reliability function or survival function of two components mixture of a generalized inverted exponential distribution is given by:

$$R(t) = p_1\left\{1 - exp\left(-\frac{\lambda_1}{t}\right)\right\}^{\alpha_1} + p_2\left\{1 - exp\left(-\frac{\lambda_2}{t}\right)\right\}^{\alpha_2}.$$

(b) Failure rate function

The failure rate function (hazard rate function) of the two components mixture of generalized inverted exponential distribution is given by:

$$r(t) = \frac{p_1\left(\frac{\alpha_1\lambda_1}{t^2} exp\left(-\frac{\lambda_1}{t}\right)\left\{1 - exp\left(-\frac{\lambda_1}{t}\right)\right\}^{\alpha_1 - 1}\right) + p_2\left(\frac{\alpha_2\lambda_2}{t^2} exp\left(-\frac{\lambda_2}{t}\right)\left\{1 - exp\left(-\frac{\lambda_2}{t}\right)\right\}^{\alpha_2 - 1}\right)}{p_1\left\{1 - exp\left(-\frac{\lambda_1}{t}\right)\right\}^{\alpha_1} + p_2\left\{1 - exp\left(-\frac{\lambda_2}{t}\right)\right\}^{\alpha_2}},$$

which can be written considering the result of Al-Hussaini and Sultan (2001) as the derivative of hazard rate function is given as $r(t)=h(t)r_1(t)+\{1- h(t)\}r_2(t)$. The derivative of hazard rate function is given as

$$r'(t) = h(t)r_1'(t) + \{1 - h(t)\}r_2'(t) - h(t)\{1 - h(t)\}\{r_1(t) - r_2(t)\}^2,$$

were $h(t) = \frac{1}{1 + \frac{p_2R_2(t)}{p_1R_1(t)}}$, $r_i(t) = \frac{f_i(t)}{R_i(t)}$ and $i = 1, 2.$ The failure rate function of two components mixture of a generalized inverted exponential distribution satisfies the following limits: $\lim_{t\to\infty} h(t) = \frac{p_1}{p_1 + p_2} = p_1$, $\lim_{t\to\infty} \frac{p_2R_2(t)}{p_1R_1(t)} = \frac{p_2}{p_1} \neq -1.$ and It follows that $h(t) < \infty$. It can be shown that $\lim_{t\to\infty} r_i(t) = 0$ for $i = 1, 2.$

The failure rate function of two components mixture of a generalized inverted exponential distribution increases initially, then decreases and eventually approaches to zero. This means that items with generalized inverted exponential distribution have a higher chance of failing as they survive for some period of time, but after survival to a specific age, the probability of failure decreases as time increases. The hazard rate function components and their mixtures are shown in Figure 2.

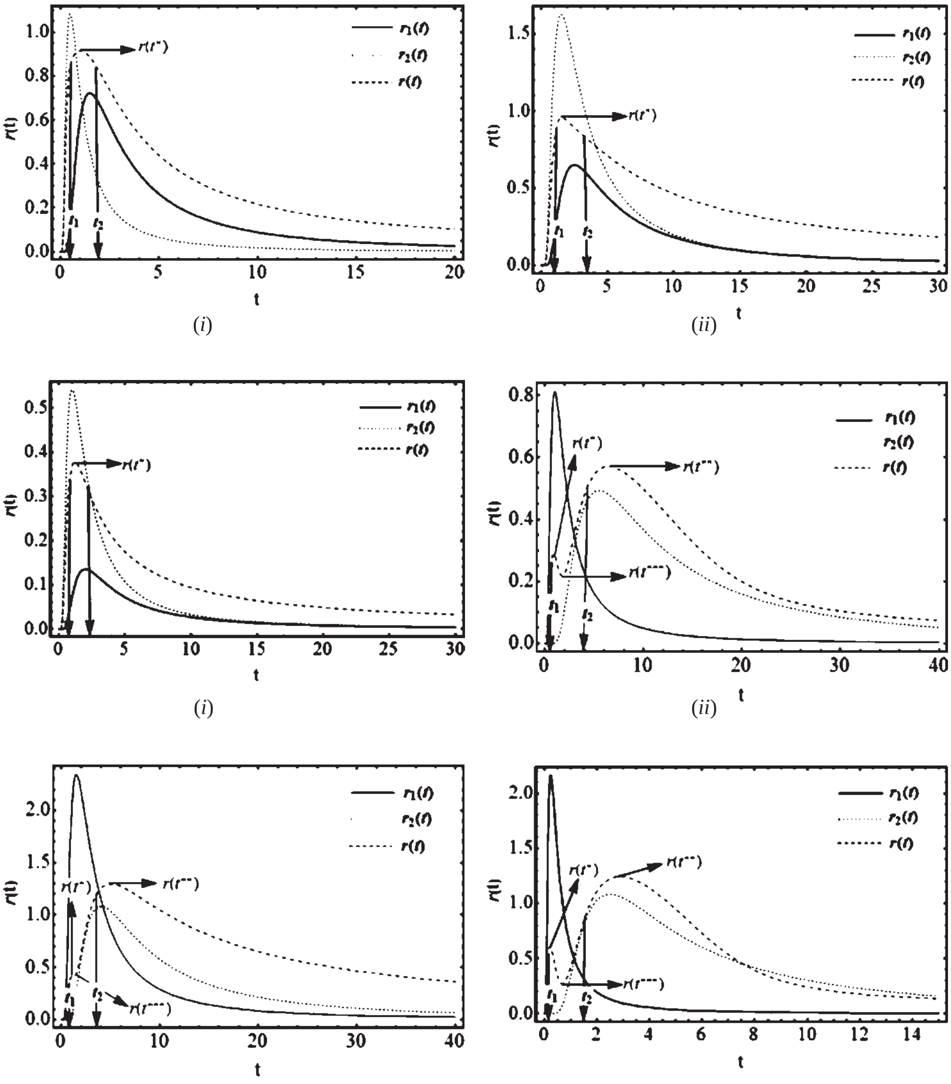


Fig. 2 Hazard rate function components and their mixtures $(p_1, \lambda_1, \lambda_2, \alpha_1, \alpha_2)$ (*i*) (0.3, 3, 1, 4, 2), (*ii*) (0.4, 5, 3, 6, 9), (*iii*) (0.4, 4, 2,1, 2), (*iv*) (0.4, 2, 11, 3, 10), (*v*) (0.3, 3, 8, 13, 16) and (*vi*) (0.3, 0.5, 5, 2, 10).

(*c*) ***Analysis of failure rate curves***: We assume that $t_1 = min\left(t_1^*, t_2^*\right)$ and $t_2 = max\left(t_1^*, t_2^*\right)$ where $t_i^*\left(i=1,2\right)$ be the mode of the density function. It is analyzed that both densities in the numerator of $r_i(t)$ enhance on $(0,t_1)$, while the denominator decreases on the same interval. Hence $r(t)$ is an increasing function on the interval $(0, t_1)$. Likewise, $r(t) \to 0$, when t approaches 0. Two cases exist within the interval (t_1, ∞) named as (*i*) Unimodal and (*ii*) Bimodal.

(*i*) ***Unimodal case***: Suppose that the maximum point of the failure rate mixturet is t^*. The difference between $r_1(t)$ and $r_2(t)$ on the interval (t_1, t^*) is sufficiently small that the first two terms of $r'(t)$ influence the third term and hence $r'(t)>0$ on the aforementioned interval. Overall the failure rate of the mixture model is an increasing function on $(0, t^*)$ and a decreasing function on (t^*, ∞) and approaches zero when $t \to 0$ Figs. 2(*i-iii*).

(*ii*) ***Bimodal case***: The smaller and larger maximum points of the failure rate mixture are denoted by t^*, and t^{**} respectively. The failure rate mixture model is an increasing function on the intervals $(0, t^*)$ and (t^{***}, t^{**}), while it decreases on the intervals (t^*, t^{***}) and (t^{**}, ∞), tends to zero as $t \to \infty$, Figs. 2(*iii-vi*).

(*d*) ***Median and Mode***: The median and mode of the two components mixture of generalized inverted exponential distributionare developed by solving the nonlinear equation with respect to t.

$$p_1\left(1-\left\{1-exp\left(-\frac{\lambda_1}{t}\right)\right\}^{\alpha_1}\right)+p_2\left(1-\left\{1-exp\left(-\frac{\lambda_2}{t}\right)\right\}^{\alpha_2}\right)=0.5.$$

$$\sum_{i=1}^{2}\left\{1-exp\left(-t^{-1}\lambda_i\right)\right\}^{\alpha_i}\frac{p_i\lambda_i\alpha_i}{\left(x^2\left\{-1+exp\left(t^{-1}\lambda_i\right)\right\}\right)^3}\begin{pmatrix}6t^2-6t\lambda_i\alpha_i+\lambda_i^2\alpha_i^2+exp\left(2t^{-1}\lambda_i\right)\left\{6t^2-6t\lambda_i+\lambda_i^2\right\}\\+exp\left(t^{-1}\lambda_i\right)\left\{-12t^2+6t\left(1+\alpha_i\right)\lambda_i+\left(1-3\alpha_i\right)\lambda_i^2\right\}\end{pmatrix}=0.$$

Tab. 1: Mean median and mode for the two-component mixture of a generalized inverted exponential distribution.

$(p_1, \lambda_1, \lambda_2, \alpha_1, \alpha_2)$	Mean	Median	Mode	$(p_1, \lambda_1, \lambda_2, \alpha_1, \alpha_2)$	Mean	Median	Mode
0.2, 3, 1, 4, 2	1.51679	0.96429	0.73278	0.2, 2, 11, 3, 10	3.91200	3.64310	0.817301, 3.33153
0.4, 3, 1, 4, 2	1.64729	1.14314	0.83342	0.4, 2, 11, 3, 10	3.36552	3.05082	0.81462, 3.14220
0.6, 3, 1, 4, 2	1.77779	1.32297	0.85582	0.6, 2, 11, 3, 10	2.81905	2.20966	0.813773, 2.00589
0.2, 5, 3, 6, 9	1.54266	1.27953	0.98166	0.2, 0.5, 5, 2, 10	1.75989	1.65276	1.53242, 2.39757
0.4, 5, 3, 6, 9	1.81044	1.46550	1.02989	0.4, 0.5, 5, 2, 10	1.49321	1.36582	1.48871, 2.94723
0.6, 5, 3, 6, 9	2.07823	1.71812	1.14851	0.6, 0.5, 5, 2, 10	1.22652	0.87747	1.37289, 3.49239

The parametric values $(p_1, \lambda_1, \lambda_2, \alpha_1, \alpha_2)$ in Table 1 are chosen to show the unimodal and bimodal cases for the mixture density function for some parameter values. The increasing order of mean, median and mode are observe in the unimodal case when the mixing proportion parameter p_1 increases. On the contrary, for bimodal case when mixing proportion parameter p_1 increases, inverse behavior has been noted for mean, median and mode.

Suppose the *n* units from the above cited mixture model are used in life testing experiment with a fixed test termination time *T*. After the test has been performed, it is observed that out of *n* units, *r* units have failed till the test termination time *T*, while *n-r* units are still working. Following the sampling scheme proposed by Mendenhall and Hader (1958), in many real life situations only the failed items can be identified as the members of the first and the second subpopulation respectively. Here it is clear that $r = r1 + r2$ and the remaining n-r units that are still functioning provide no information about the population to which they belong. Let *xij* be defined as the failure time of the jth unit from ith subpopulation, where $j = 1, 2\ldots, r_i$, $i = 1, 2$, $0 < x_{1j}, x_{2j} \le T$.

$$L(\Theta \mid \mathrm{x}) \propto \left\{\prod_{j=1}^{r_1} p_1 f_1\left(x_{1j}\right)\right\}\left\{\prod_{j=1}^{r_2} p_2 f_2\left(x_{2j}\right)\right\}\{1-F(T)\}^{n-r}$$

$$L(\Theta \mid \mathrm{x}) \propto \prod_{j=1}^{r_1}\left(p_1 \frac{\alpha_1\lambda_1}{x_{1j}^2} exp\left[-\frac{\lambda_1}{x_{1j}}\right]\left\{1-exp\left[-\frac{\lambda_1}{x_{1j}}\right]\right\}^{\alpha_1-1}\right)\prod_{j=1}^{r_2}\left(p_2 \frac{\alpha_2\lambda_2}{x_{2j}^2} exp\left[-\frac{\lambda_2}{x_{2j}}\right]\left\{1-exp\left[-\frac{\lambda_2}{x_{2j}}\right]\right\}^{\alpha_2-1}\right)$$
$$\times\left\{1-p_1\left\{1-\left(1-exp\left[-\frac{\lambda_1}{T}\right]\right)^{\alpha_1}\right\}-p_2\left\{1-\left(1-exp\left[-\frac{\lambda_2}{T}\right]\right)^{\alpha_2}\right\}\right\}^{(n-r)},$$

$$L(\Theta \mid \mathrm{x}) \propto \sum_{k=0}^{n-r}\binom{n-r}{k} p_1^{r_1+k}(\alpha_1\lambda_1)^{r_1} exp\left\{-\lambda_1\sum_{i=1}^{r_1}\left(\frac{1}{x_{1j}}\right)\right\} exp\left\{-(\alpha_1-1)\sum_{j=1}^{r_2} ln\left(1-exp\left(-\frac{\lambda_1}{x_{1j}}\right)\right)\right\}\Delta_1(\Phi_1)$$
$$\times p_2^{n-r_1-k}\,(\alpha_2\lambda_2)^{r_2} exp\left\{-\lambda_2\sum_{j=1}^{r_2}\left(\frac{1}{x_{2j}}\right)\right\} exp\left\{-(\alpha_2-1)\sum_{j=1}^{r_2} ln\left(1-exp\left(-\frac{\lambda_2}{x_{2j}}\right)\right)\right\}\Delta_2(\Phi_2). \tag{2}$$

The likelihood function has the following form:

Assuming the shape parameter to be known, the likelihood function (2) reduces to

$$L(\Delta \mid \xi) \propto \sum_{k=0}^{n-r}\binom{n-r}{k} p_1^{r_1+k}(\lambda_1)^{r_1} exp\left\{-\lambda_1\sum_{i=1}^{r_1}\left(\frac{1}{x_{1j}}\right)\right\} exp\left\{-(\alpha_1-1)\sum_{j=1}^{r_1} ln\left(1-exp\left(-\frac{\lambda_1}{x_{1j}}\right)\right)\right\}\Delta_1(\Phi_1)$$
$$\times p_2^{n-r_1-k}\,(\lambda_2)^{r_2} exp\left\{-\lambda_2\sum_{j=1}^{r_2}\left(\frac{1}{x_{2j}}\right)\right\} exp\left\{-(\alpha_2-1)\sum_{j=1}^{r_2} ln\left(1-exp\left(-\frac{\lambda_2}{x_{2j}}\right)\right)\right\}\Delta_2(\Phi_2),$$

where $\Delta = (\lambda_1, \lambda_2, p_1)$, $\Delta_1(\Phi_1) = exp\left\{\alpha_1 k\, ln\left(1-exp\left(-\frac{\lambda_1}{T}\right)\right)\right\}$ (3)

$$\Delta_2(\Phi_2) = exp\left\{\alpha_2(n-r_1-k)\, ln\left(1-exp\left(-\frac{\lambda_2}{T}\right)\right)\right\}.$$

3. BAYESIAN ESTIMATION OF PARAMETERS

In this section, we discuss prior distributions for unknown parameters, loss functions and Bayes estimators and their posterior risks.

3.1 BAYESIAN ESTIMATION USING INFORMATIVE PRIOR

The Bayesian analysis requires the choice of suitable priors for the unknown parameters in addition to the experimental data. The main objective in this bond is the relationship between the prior distribution and the loss function. The mixture model under consideration has two scale parameters and one mixing proportion parameter. We consider both the informative and noninformative priors. First, we assume that scale parameter λ_i, $i = 1, 2$, has independent gamma priors with the shape and scale parameters as a_i, and b_i respectively, $g(\lambda_i \mid a_i, b_i) \propto \lambda_i^{(a_i-1)}$, exp $(-b_i\lambda_1)$ and uniform prior for By combining the joint prior with likelihood function given in (3) we obtain the following joint posterior distribution of λ_i, p_1 as:

$$g(\Delta \mid \mathrm{x}) = \frac{\sum_{k=0}^{n-r}\binom{n-r}{k} p_1^{r_1+k} p_2^{n-r_1-k} \prod_{i=1}^{2} \lambda_i^{\{r_i+(a_i-1)\}} exp\left(-\left(\lambda_i(\Lambda_i+b_i)+(\alpha_i-1)\psi_i\right)\right)\left\{\Delta_1(\Phi_1)\right\}\left\{\Delta_2(\Phi_2)\right\}}{\sum_{k=0}^{n-r}\binom{n-r}{k} B(r_1+k+1, n-r_1-k+1)\int_0^\infty\int_0^\infty \prod_{i=1}^{2} \lambda_i^{\{r_i+(a_i-1)\}} exp\left(-\left(\lambda_i(\Lambda_i+b_i)+(\alpha_i-1)\psi_i\right)\right)\left\{\Delta_1(\Phi_1)\right\}\left\{\Delta_2(\Phi_2)\right\} d\lambda_i}$$

where $\Lambda_i = \sum_{i=1}^{r_i}\left(\frac{1}{x_{ij}}\right), \psi_i = \sum_{j=1}^{r_i} ln\left(1-exp\left(-\frac{\lambda_i}{x_{ij}}\right)\right), \Delta_1(\Phi_1) = exp\left\{\alpha_1 k\, ln\left(1-exp\left(-\frac{\lambda_1}{T}\right)\right)\right\}$

$$\Delta_2(\Phi_2) = exp\left\{\alpha_2(n-r_1-k)\, ln\left(1-exp\left(-\frac{\lambda_2}{T}\right)\right)\right\}, \; i = 1, 2.$$

The marginal distribution of λ_1 is simply the probability distribution of λ_1 that neglects other nuisance information about λ_2 and p_1 which is obtained by integrating the joint probability distribution with respect to other parameters as:

$$g(\lambda_1 \mid \mathrm{x}) = \frac{\begin{array}{c}\sum_{k=0}^{n-r}\binom{n-r}{k} B(r_1+k+1, n-r_1-k+1)\lambda_1^{\{r_1+(a_1-1)\}} exp\left(-\left(\lambda_1(\Lambda_1+b_1)+(\alpha_1-1)\psi_1\right)\right)\left\{\Delta_1(\Phi_1)\right\} \\ \times\int_0^\infty \lambda_2^{\{r_2+(a_2-1)\}} exp\left(-\left(\lambda_2(\Lambda_2+b_2)+(\alpha_2-1)\psi_2\right)\right)\left\{\Delta_2(\Phi_2)\right\} d\lambda_2\end{array}}{\sum_{k=0}^{n-r}\binom{n-r}{k} B(r_1+k+1, n-r_1-k+1)\int_0^\infty\int_0^\infty \prod_{i=1}^{2} \lambda_i^{\{r_i+(a_i-1)\}} exp\left(-\left(\lambda_i(\Lambda_i+b_i)+(\alpha_i-1)\psi_i\right)\right)\left\{\Delta_1(\Phi_1)\right\}\left\{\Delta_2(\Phi_2)\right\} d\lambda_i},$$

Similarly, the marginal posterior distribution of λ_2 and p_1 are derived as:

$$g(\lambda_2 \mid \mathrm{x}) = \frac{\begin{array}{c}\sum_{k=0}^{n-r}\binom{n-r}{k} B(r_1+k+1, n-r_1-k+1)\lambda_2^{\{r_2+(a_2-1)\}} exp\left(-\left(\lambda_2(\Lambda_2+b_2)+(\alpha_2-1)\psi_2\right)\right)\left\{\Delta_2(\Phi_2)\right\} \\ \times\int_0^\infty \lambda_1^{\{r_1+(a_1-1)\}} exp\left(-\left(\lambda_1(\Lambda_1+b_1)+(\alpha_1-1)\psi_1\right)\right)\left\{\Delta_1(\Phi_1)\right\} d\lambda_1\end{array}}{\sum_{k=0}^{n-r}\binom{n-r}{k} B(r_1+k+1, n-r_1-k+1)\int_0^\infty\int_0^\infty \prod_{i=1}^{2} \lambda_i^{\{r_i+(a_i-1)\}} exp\left(-\left(\lambda_i(\Lambda_i+b_i)+(\alpha_i-1)\psi_i\right)\right)\left\{\Delta_1(\Phi_1)\right\}\left\{\Delta_2(\Phi_2)\right\} d\lambda_i},$$

3.2 BAYESIAN ESTIMATION OF THE MIXTURE MODEL ASSUMING THE NONINFORMATIVE PRIORS

The noninformative priors are a significant part of a Bayesian tool kit. The noninformative priors have a limited effect on the ultimate inference comparative to the data. Bernardo (1979) contended that a noninformative prior should be considered as a reference prior, i.e., a prior that is favourable for use as a standard when scrutinize statistical data. The most common example of noninformative prior is uniform prior that is employed when no conventional prior information is available.

3.2.1 POSTERIOR DISTRIBUTION USING UNIFORM PRIOR

The uniform prior for the unknown parameter λ_i can be written as λ_i:*Uniform* $(0, \infty)$, $i = 1,2$. We suppose a priori that (λ_i, p_1) are independent and also assume that p_1~*Uniform* (0,1). Thus the joint prior distribution of (λ_i, p_1) is $p(\lambda_i, p_1) \propto k$. We obtain the joint posterior distribution merging the likelihood function given in (3) with uniform prior information as:

$$g(\Delta \mid \mathrm{x}) = \frac{\sum_{k=0}^{n-r} \binom{n-r}{k} p_1^{\,r_1+k} p_2^{\,n-r_1-k} \prod_{i=1}^{2} \lambda_i^{r_i}\, exp\left(-\left(\lambda_i(\Lambda_i) + (\alpha_i - 1)\psi_i\right)\right)\left\{\Delta_1(\Phi_1)\right\}\left\{\Delta_2(\Phi_2)\right\}}{\sum_{k=0}^{n-r} \binom{n-r}{k} B(r_1+k+1, n-r_1-k+1) \int_0^\infty \int_0^\infty \prod_{i=1}^{2} \lambda_i^{r_i}\, exp\left(-\left(\lambda_i(\Lambda_i) + (\alpha_i - 1)\psi_i\right)\right)\left\{\Delta_1(\Phi_1)\right\}\left\{\Delta_2(\Phi_2)\right\} d\lambda_i}.$$

Marginal distributions of λ_i and $p_1$can be obtained by hazardous nuisance parameters. For space restriction, we do not present the expression for the marginal distributions under noninformative priors.

3.3 BAYESIAN ESTIMATORS UNDER DIFFERENT LOSS FUNCTIONS

In order to take an optimum decision, a suitable loss function must be specified. The choice of a loss function is a difficult job: its selection is often based on the reasons of mathematical convenience without any particular reason of ongoing interest excluding cost effect. As in risk analysis, the potentiality of undesired events and its consequences are explored. This potentiality is usually measured through failure rate. In disastrous outcomes, it can be difficult to underestimate the potentiality of an event rather than to overestimate it. This is significant when the risk level is the basis of a risk reducing initiative, either by reducing the potentiality or the consequences of the event. It is unreasonable to use a loss function that allows the estimation of a failure probability of zero. A positive loss at the origin allows the estimation of zero and in risk analysis estimating a zero failure probability simply means that no risk is expected (for further details see Norstrom (1996)). Five loss

functions are used to obtain the Bayes estimators along with posterior risks, i.e.,the squared error (SE) loss function, weighted squared error (WSE) loss function,the precautionary (P) loss function and quadratic (Q) loss function, modified squared error (MSE) loss function. The most commonly used loss function is (SE) loss function defined by $L_1 = \left(\hat{\theta}_{SE} - \theta\right)^2$,where $\hat{\theta}_{SE}$ is a decision rule to estimate parameter θ. The Bayes estimator under SE loss function is $\hat{\theta}_{SE} = E(\theta \mid \mathrm{x})$ and posterior risk under SE loss function is $\rho\left(\hat{\theta}_{SE}\right) = E\left(\theta^2 \mid \mathrm{x}\right) - \left\{E\left(\theta \mid \mathrm{x}\right)\right\}^2$. The weighted squared error (WSE) loss function which is of concern is $L_2 = \theta^{-1}\left(\theta - \hat{\theta}_{WSE}\right)^2$, the Bayes estimator under WSE loss function is $\hat{\theta}_{WSE} = \left\{E\left(\theta^{-1} \mid \mathrm{x}\right)\right\}^{-1}$ and posterior risk under WSE loss function is $\rho\left(\hat{\theta}_{WSE}\right) = E(\theta \mid \mathrm{x}) - \left\{E\left(\theta^{-1} \mid \mathrm{x}\right)\right\}^{-1}$. Norstrom (1996) has introduced a precautionary loss function and is defined as $L_3 = \left(\theta - \hat{\theta}_p\right)^2 \hat{\theta}_p^{\,-1}$, where $\hat{\theta}_p$ is a decision rule to estimate parameter θ. The Bayes estimator under P loss function is $\hat{\theta}_p = \sqrt{E\left(\theta^2 \mid \mathrm{x}\right)}$ and posterior risk under P loss function is $\rho\left(\hat{\theta}_p\right) = 2\left\{\sqrt{E\left(\theta^2 \mid \mathrm{x}\right)} - E(\theta \mid \mathrm{x})\right\}$. The quadratic loss function which is defined as $L_4 = \left(1 - \theta^{-1}\hat{\theta}_Q\right)^2$ the Bayes estimator and posterior risk under Q loss function are $\hat{\theta}_Q = \left\{E\left(\theta^{-2} \mid \mathrm{x}\right)\right\}^{-1} E\left(\theta^{-1} \mid \mathrm{x}\right)$, $\rho\left(\hat{\theta}_Q\right) = 1 - \left\{E\left(\theta^{-2} \mid \mathrm{x}\right)\right\}^{-1}\left\{E\left(\theta^{-1} \mid \mathrm{x}\right)\right\}^2$.
The modified squared error (MSE) loss function was introduced by Degroot (1970), which is of concern is $L_5 = \theta^{-2}\left(\theta - \hat{\theta}_{MSE}\right)^2$. The Bayes estimator under MSE loss function is $\hat{\theta}_{MSE} = \left\{E(\theta \mid \mathrm{x})\right\}^{-1}\left\{E\left(\theta^2 \mid \mathrm{x}\right)\right\}^{-1}$ and posterior risk under MSE loss function is $\rho\left(\hat{\theta}_{MSE}\right) = 1 - \left\{E(\theta \mid \mathrm{x})\right\}^{-1}\left\{E(\theta \mid \mathrm{x})\right\}^2$, where E denotes the expectation with respect to the posterior distribution of θ. Thus the posterior expectation of any function of parameter, say $U\left(\lambda_1, \lambda_2, p_1\right)$ can be written as:

$$\hat{U}(\lambda_1, \lambda_2, p_1) = E\left\{U(\lambda_1, \lambda_2, p_1) \mid (\mathrm{x})\right\} = \frac{\int_0^\infty \int_0^\infty \int_0^1 U(\lambda_1, \lambda_2, p_1) g(\lambda_1, \lambda_1, p_1 \mid \mathrm{x}) dp_1 d\lambda_1 d\lambda_2}{\int_0^\infty \int_0^\infty \int_0^1 g(\lambda_1, \lambda_1, p_1 \mid \mathrm{x}) dp_1 d\lambda_1 d\lambda_2}. \quad (4)$$

However, it is not possible to evaluate estimates of set of parameters analytically. The estimates are not in a closed form and therefore must be evaluated numerically.

4. SIMULATION AND COMPARISON OF THE ESTIMATORS

In this section, a Monte-Carlo simulation study is conducted to analyze the behavior of the proposed estimators for different sample sizes, different priors, different parametric values $(\lambda_1, \lambda_2) \in (1, 3), (4, 2), T = 30, p_1 \in 0.3.$ Samples of size n = 30, 60 and 90 were generated from the two components mixture of generalized inverted exponential distribution. A well-known procedure in simulation for computer generation of random variables is the inverse transform method. This method provides the most straight forward procedure to generate samples of a given distribution when its quantile function exists in closed form. Probabilistic mixing is used to generate the mixture data. To generate the mixture model, a random number 'u' is generated from the uniform distribution on (0, 1). If $u < p_1$ the observation is taken randomly from F_1 (the generalized inverted exponential distribution with parameter λ_1) and if $u > p_1$ the observation is taken randomly from F_2(the generalized inverted exponential distribution with parameter λ_2). The values of hyperparameters (a_1, b_1, a_2, b_2) have been selected in such a manner that the prior mean becomes the expected value of the corresponding parameter.The hyperparameters considered in the simulation study are (3, 3, 6, 2) and (8, 2, 6, 3). All observations that exceed T are treated as censored. For each of the combinations of parameters, sample sizes, we generated 1000 samples using Mathematica. For each of 1000 samples, the average of these estimates and corresponding posterior risks are reported in Tables 2 to 6.

Tab. 2: Bayes estimates and their posterior risks in parentheses under SE loss function for $\alpha_1 = \alpha_2 = 0,5$

$\Delta = (\lambda_1, \lambda_2, p_1)$		*Uniform prior*			*Gamma prior*		
(1, 3, 0.3)	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
	30	1.41443 (0.36065)	3.25745 (0.61236)	0.33289 (0.22065)	1.19979 (0.17501)	2.89802 (0.54072)	0.32238 (0.00833)
	60	1.22456 (0.16124)	3.12897 (0.38122)	0.31065 (0.21319)	1.18172 (0.12254)	2.93128 (0.32102)	0.32171 (0.00454)
	90	1.10321 (0.13380)	3.12413 (0.28908)	0.30388 (0.21091)	1.03935 (0.06873)	3.06832 (0.26240)	0.31354 (0.00293)
(4, 2, 0.3)	30	3.45942 (0.96688)	2.3256 (0.43184)	0.32558 (0.21705)	3.66342 (0.63912)	2.19192 (0.29419)	0.29943 (0.00888)
	60	3.76376 (0.64305)	2.02452 (0.19458)	0.32277 (0.21682)	3.73881 (0.51512)	1.96942 (0.15062)	0.31083 (0.00502)
	90	3.82385 (0.57066)	2.01386 (0.17440)	0.29204 (0.20557)	3.89622 (0.44599)	1.97576 (0.10626)	0.30057 (0.00338)

Tab. 3: Bayes estimates and their posterior risks in parentheses under WSE loss function for $\alpha_1 = \alpha_2 = 0,5$

$\Delta = (\lambda_1, \lambda_2, p_1)$	*Uniform prior*				*Gamma prior*		
(1, 3, 0.3)	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
	30	1.27804 (0.23421)	2.95408 (0.21022)	0.32255 (0.02852)	0.90422 (0.12184)	2.92654 (0.17127)	0.29272 (0.02816)
	60	1.18654 (0.12259)	2.96841 (0.12294)	0.31169 (0.01425)	0.99252 (0.08906)	3.16251 (0.11014)	0.29292 (0.01467)
	90	1.11736 (0.08421)	3.03446 (0.09202)	0.29601 (0.01209)	0.99862 (0.06392)	3.10927 (0.08017)	0.30449 (0.00975)
(4, 2, 0.3)	30	3.15788 (0.37712)	2.39382 (0.20534)	0.27225 (0.03306)	3.33275 (0.20673)	1.53806 (0.13504)	0.26159 (0.03158)
	60	3.27379 (0.25784)	2.11814 (0.100261)	0.28028 (0.01724)	3.53594 (0.15596)	1.98731 (0.07414)	0.28327 (0.01722)
	90	3.48172 (0.19301)	1.97915 (0.06239)	0.28923 (0.01199)	3.71968 (0.12858)	2.02962 (0.05543)	0.29855 (0.01151)

Tab. 4: Bayes estimates and their posterior risks in parentheses under P loss function for $\alpha_1 = \alpha_2 = 0,5$

$\Delta = (\lambda_1, \lambda_2, p_1)$	*Uniform prior*				*Gamma prior*		
(1, 3, 0.3)	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
	30	1.44482 (0.20249)	3.46408 (0.18329)	0.33586 (0.02587)	1.19865 (0.14559)	3.09169 (0.18119)	0.32046 (0.02543)
	60	1.30518 (0.11689)	3.25899 (0.11812)	0.32547 (0.01394)	1.19724 (0.09291)	3.06531 (0.11409)	0.31813 (0.01380)
	90	1.22920 (0.07862)	3.22525 (0.08860)	0.31568 (0.00939)	1.12369 (0.07364)	3.02841 (0.08014)	0.31159 (0.00926)
(4, 2, 0.3)	30	3.62231 (0.26817)	2.45215 (0.17910)	0.32801 (0.02894)	3.67975 (0.17843)	2.22254 (0.12847)	0.32645 (0.02853)
	60	3.85142 (0.18575)	2.36166 (0.09946)	0.31468 (0.01637)	3.73249 (0.13603)	2.10748 (0.07212)	0.30922 (0.01623)
	90	4.01127 (0.13375)	2.14132 (0.06191)	0.30302 (0.01113)	4.02511 (0.10626)	2.10239 (0.05874)	0.30865 (0.01110)

The analysis of Tables 2 to 6, leads to the following conclusions: The foremost point that requires attention is that the estimated risks of estimatorsdecrease as sample size increases. Bayesian estimates become very close to the true values of the parameters as we increase the sample size. With a large parametric value, the corresponding posterior risk is a high. Bayes estimators performed well under the mean squared error loss function than the rest loss functions. Bayes estimates are found to be underestimated under Q loss function based on both priors. Bayes estimators are efficient under MSE loss function; In fact, the use of mean squared error loss function unveiled the smallest posterior risk, which is really an advantageous property. Furthermore, we obtain efficient results using the gamma prior than the uniform prior. Posterior risks for the Bayes estimates assuming uniform prior is also little high. Hence, gamma prior has a clear edge over uniform prior, and this allows us making a selection of preferable priors and loss functions.

Tab. 5: Bayes estimates and their posterior risks in parentheses under Q loss function for $\alpha_1 = \alpha_2 = 0{,}5$

$\Delta = (\lambda_1, \lambda_2, p_1)$		*Uniform prior*			*Gamma prior*		
(1, 3, 0.3)	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
	30	1.42198 (0.15933)	2.71533 (0.08324)	0.28587 (0.10372)	0.88985 (0.12451)	2.57169 (0.06229)	0.27185 (0.10158)
	60	1.19756 (0.09666)	3.04609 (0.04324)	0.29019 (0.05033)	0.89313 (0.08541)	2.80753 (0.03678)	0.28432 (0.05082)
	90	0.96084 (0.07182)	3.04502 (0.02867)	0.29881 (0.03387)	0.92031 (0.06915)	3.09514 (0.02599)	0.29752 (0.03622)
(4, 2, 0.3)	30	2.63941 (0.16928)	1.85095 (0.09359)	0.23829 (0.13876)	3.32534 (0.06399)	1.98038 (0.06659)	0.25039 (0.13328)
	60	3.05887 (0.09285)	2.01323 (0.04771)	0.25889 (0.06833)	3.43282 (0.05256)	1.98509 (0.03841)	0.26126 (0.06552)
	90	3.40641 (0.05712)	2.05516 (0.03229)	0.28698 (0.04011)	3.56395 (0.03967)	1.98966 (0.02798)	0.29817 (0.04249)

Tab. 6: Bayes estimates and their posterior risks in parentheses under MSE loss function for $\alpha_1 = \alpha_2 = 0{,}5$

$\Delta = (\lambda_1, \lambda_2, p_1)$		*Uniform prior*			*Gamma prior*		
(1, 3, 0.3)	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
	30	1.56316 (0.13983)	3.17613 (0.06539)	0.34433 (0.07919)	1.39264 (0.11074)	3.13597 (0.05137)	0.34879 (0.07387)
	60	1.20835 (0.08519)	3.13318 (0.03873)	0.31817 (0.04427)	1.29967 (0.07846)	3.12355 (0.03311)	0.32437 (0.04465)
	90	1.13228 (0.06587)	3.12494 (0.02648)	0.31605 (0.03102)	1.06324 (0.05819)	3.11485 (0.02459)	0.31339 (0.03011)
(4, 2, 0.3)	30	3.81075 (0.07662)	2.72958 (0.07206)	0.33941 (0.09331)	3.75514 (0.04864)	2.28751 (0.05724)	0.33764 (0.09217)
	60	3.87797 (0.05508)	2.31112 (0.04341)	0.32772 (0.04859)	3.94932 (0.03452)	2.18939 (0.03619)	0.32983 (0.04722)
	90	3.87857 (0.04647)	2.24802 (0.02984)	0.31217 (0.03695)	4.00175 (0.02892)	2.03815 (0.02642)	0.31931 (0.03704)

5. APPLICATION

In this section, we analyze real data set to illustrate the methodologies discussed in the previous sections. The data set represents the lifetimes of 50 devices. The data, obtained from Aarset (1987), are given below:

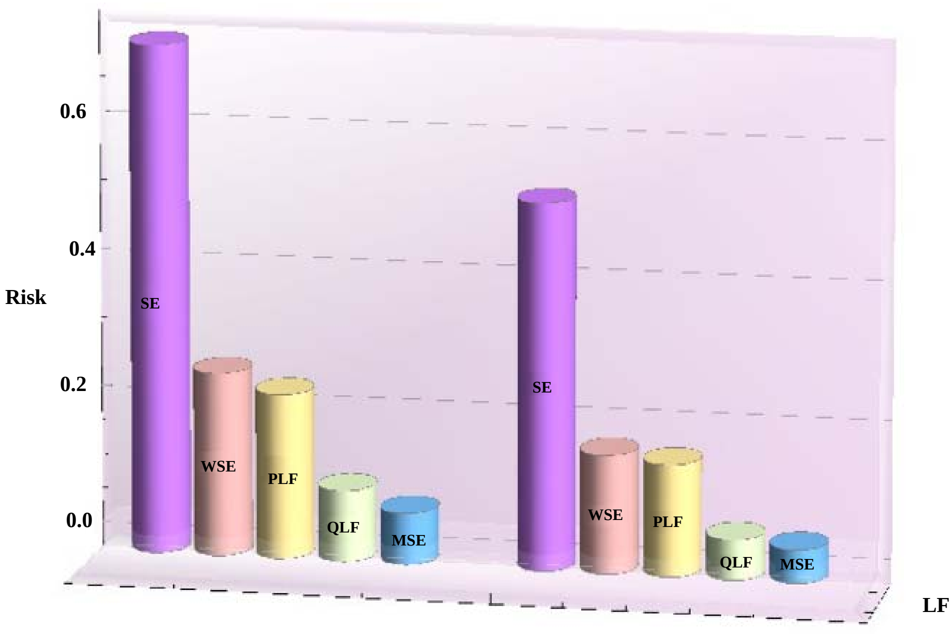
1, 7, 18, 40, 45, 50, 55, 86, 85, 85, 85, 84, 84, 84, 79, 75, 72, 67, 67, 63, 60, 18, 11, 6, 3, 2, 1, 1, 0.2, 0.1, 1, 18, 86, 85, 85, 83, 82, 82, 67, 67, 63, 47, 46, 36, 32, 21, 18, 12, 7, 1.

Among the 50 observations, the two observations, i.e., 0.2, 0.1 considered as outliers, are discarded. The generalized inverted exponential distribution is considered as a suitable candidate for modeling complex lifetime data sets, so we can employ this data to the generalized inverted exponential mixture model. Now we assume that when a failure occurs, we can identify the object as per its cause of failure and regard it as belonging to population I or population II, respectively. We have taken $n_1 = 20$, $n_2 = 28$, $r_1 = 16$ and $r_2 = 24$. The following information is extracted for our mixture modelby taking censoring time $T = 84$, $n = 48, r = 40, \sum_{j=1}^{r_1} x_{1j}^{-1} = 3.78676$ and $\sum_{j=1}^{r_2} x_{2j}^{-1} = 3.36858$. Bayes estimates are obtained by assumingboth priors using the informative and non-informative priors under five loss functions. Bayes estimates and posterior risks for real data set are listed in Table 7. It is clear that the best estimates are those with the minimum posterior risks and optimal estimates obtained under MSE loss function.

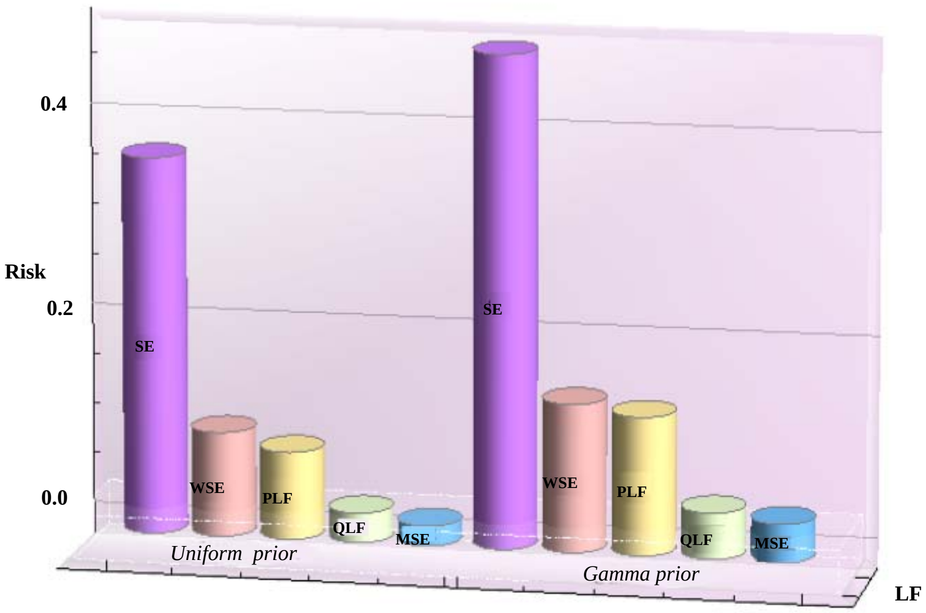
Tab. 7: Bayes estimates and their posterior risks in parentheses for real dataset.

Loss functions	*Uniform prior*			*Gamma prior*		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SE	3.02441 (0.72961)	4.27720 (0.37123)	0.38083 (0.00523)	3.23084 (0.52619)	3.51715 (0.48457)	0.38508 (0.00521)
WSE	2.76135 (0.26305)	4.17425 (0.10295)	0.366485 (0.01435)	3.06016 (0.17068)	3.37101 (0.14614)	0.37075 (0.01432)
P	3.14271 (0.23661)	4.32038 (0.08636)	0.38764 (0.01360)	3.31127 (0.16086)	3.58538 (0.13645	0.39187 (0.01357)
Q	2.47957 (0.10204)	4.05193 (0.02930)	0.35147 (0.04098)	2.88315 (0.05784)	3.21765 (0.04549)	0.35574 (0.04048)
MSE	3.26564 (0.07387)	4.36399 (0.01989)	0.39456 (0.03478)	3.39371 (0.04799)	3.65493 (0.03769)	0.39877 (0.03434)

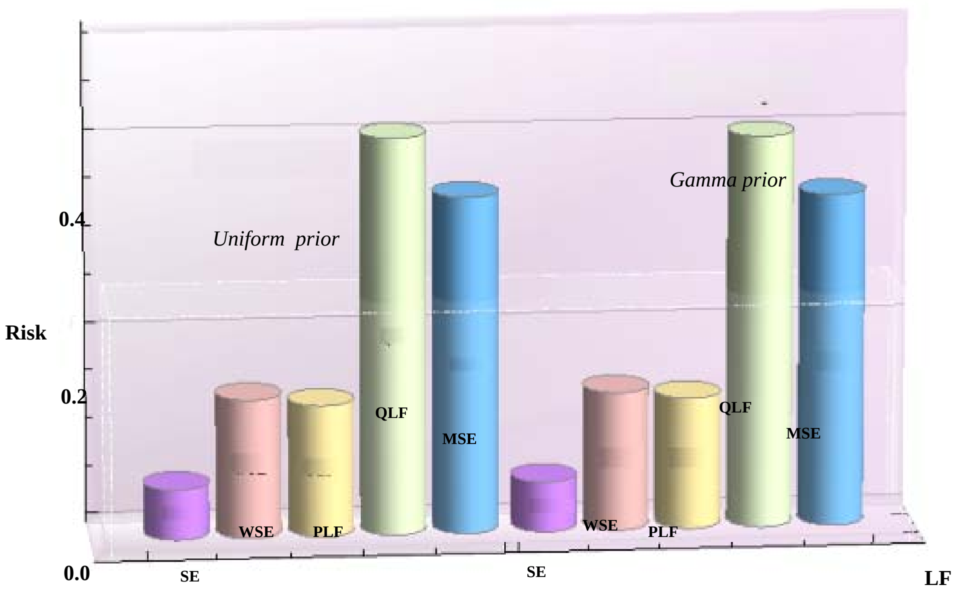
It is obvious that results obtained through real data are well-matched with simulation results, though there are some exceptions when using large data set. The Table 7 also reveals that performance of the gamma prior is best. Some graphical representation of loss functions of estimates under different priors for the mixture components are presented in Figure 3. It is manifest that the priors (gamma and uniform) influence the magnitude of the loss functions. These graphs illustrate the versatility of the loss functions under both priors in addition to a noticeable minimum magnitude of MSE loss function for the estimates of both components of mixture distribution. Due to minimum magnitude of the posterior risks the estimate of mixing proportion component SE performs better under both priors.



(a)



(b)



(c)

Fig. 3: The loss functions of estimates under uniform and gamma priors. (a) for first component $\hat{\lambda}_1$, (b) for second component $\hat{\lambda}_2$, (c) for mixing proportion component $\hat{p}_1$

6. CONCLUDING REMARKS

In this study, we propose a mixture of two-components generalized inverted exponential distribution model of lifetime study. We have discussed nice properties and estimation of parameters of the mixturedistribution using five different loss functions under informative and noninformative priors. The Bayesian analysis ensures us to perform a comprehensive selection of the suitable prior and a desirable loss function for the mixture two-components generalized inverted exponential distributions. The simulation study has revealed some interesting results related to Bayes estimates of parameters. The posterior risks of the estimates of the parameters appeared to be quite large with relatively large values of the parameters and vice versa. To address the problem of selecting prior and loss function we have observed that the Bayes estimators of parameters perform well under mean squared loss function assuming gamma prior. Practitioners can use results developed in this study to analyze heterogeneous data under Type I censoring. This newly proposed mixture distribution is worth investigating because it has the flexibility to accommodate different shapes for different values of parameters that occur in survival analysis.This work can be extended in future using different censoring schemes.

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